BOOK REVIEWS

Theorie der Funktionen einer reellen Veränderlichen. By I. P. Natanson. Berlin, Akademie-Verlag, 1954. 12+478 pp. 26 DM.

This is a textbook in real function theory, first published in 1941 in Russian. A revised and expanded Russian edition appeared in 1950, and it is this that has now been translated into German.

Briefly, the contents are as follows. (These phrases are the reviewer's summaries, not necessarily the author's chapter titles.) I–II: countable sets, open and closed sets on the real line. III–VI: Lebesgue measure on the real line, measurable functions, the Lebesgue integral and its properties. VII: orthogonality in Hilbert space, Fourier series. VIII: functions of bounded variation, and Stieltjes integrals. IX: absolute continuity and the differentiation of the Lebesgue integral. X: singular integrals and trigonometric series. XI–XIII: measure and integration in two dimensions, Fubini theorem, differentiation of set functions. XIV: ordinal numbers. XV: Baire classification of functions. XVI: topological properties of certain function spaces. XVII: the role played by the Russians in the development of real function theory.

The assumed prerequisite for a course based on this book is differential and integral calculus—in the European, not the American, manner. Thus in most schools in this country this would have to be a second course in real function theory. Specifically, the theory of limits is assumed understood—lim, lim sup, lim inf, and the notion of uniform convergence are not even defined. Furthermore, though certain properties of the real number system are investigated, this is not done in any systematic way. For example, there is a proof of the Bolzano-Weierstrass theorem based on the nonempty intersection of a nested set of closed intervals, this latter being referred to as a "well known theorem from the theory of limits."

The reviewer would like to suggest that regardless of what background is assumed, there are two essentially independent criteria on which a real function theory book may be classed as "advanced" or "elementary." (1) The degree of abstraction and generality in the subject matter considered. (2) The thoroughness with which investigations are prosecuted—the extent to which all facets of a problem are considered and "best possible" theorems are ferreted out. Natanson's book must be rated "elementary" on the first of these counts and "advanced" on the second.

Specifically, the only measure that appears in the book is Lebesgue measure, and this is only for bounded sets. In one sense this is to be commended rather than criticized. It gives the student something concrete to work with, yet the theory for this restricted case is fairly representative. On the other hand, many applications are ruled out, and the student is given no clue as to the way in which a more general theory would differ from this restricted one.

The book exudes a spirit of, "Let's see if we can't find out some more about this," and in general Natanson has done an excellent job of rounding out each discussion on which he embarks. Examples: The discussion of topology on the real line includes decomposition theorems for open and closed sets—decomposition of open sets into components and of closed sets into perfect and countable sets. The discussion of orthogonality in Hilbert space includes an investigation of completeness of orthogonal systems and the proof that L_2 and l_2 are equivalent.

Of course, any author who tries to be thorough is going to be heckled by people who can think of theorems he left out. The reviewer is really well pleased with Natanson's thoroughness but will join the hecklers with one example. In giving criteria for passage to the limit under the integral sign, Natanson in each case states his conclusion $\lim \int_E f_n(x)dx = \int_E f(x)dx$, whereas what he actually proves is $\lim \int_E |f_n(x) - f(x)| dx = 0$. Indeed, there is the slight curiosity that "convergence in the mean" is defined to mean strong convergence in L_2; the concept in L_1 is not discussed.

The book is written with the student in mind. Concrete examples accompany the introduction of new ideas. Proofs are given in sufficient, but not burdensome, detail. There is one feature, which is probably a deliberate part of Natanson's pedagogical plan, with which the reviewer would take issue. In several places there appear repetitions of essentially the same proof to give corollaries first and the master theorem at the end. For example, it is proved, in the order indicated, that convergence in measure justifies passage to the limit under the integral sign when accompanied by (a) uniform boundedness, (b) uniform domination by an integrable function, (c) uniform absolute continuity of the integrals. Admittedly, this is only one man's opinion, but the reviewer feels that one of the great thrills in the study of real function theory comes from proving a big theorem and then seeing how many corollaries drop out easily.

Through Chapter X there are exercises collected at the ends of the chapters. For the most part these seem to be fairly substantial problems, though not unreasonably difficult. There are no exercises in Chapters XI–XVII.

There is a slip in the transfinite induction theorem. It seems to permit one to prove a proposition true for "all ordinals." Incidentally, the Burali-Forti paradox on the set of all ordinals is explained two pages before this theorem. Aside from this, the reviewer found only typographical errors—and not very many of these. The format of the book is quite pleasing.

M. E. MUNROE

Differential line geometry. By V. Hlavatý. Trans. by H. Levy. Groningen, Noordhoff, 1953. 6+10+495 pp. 22.50 Dutch florins; cloth, 25 Dutch florins.

This book was first published in Czech in 1941 and was later translated into German and published by Noordhoff in 1945. The present edition is a translation from the German edition with the author's collaboration. The translator himself has added to the text by suggesting changes in some theorems and by adding a few new ones.

The book is meant to be a definitive work on three-dimensional differential line geometry, where line geometry in 3-space is studied as point geometry on a 4-dimensional quadric in a projective point space of 5 dimensions. Klein discovered the mapping of line geometry onto the 4-dimensional quadric, and for this reason the quadric is called the Klein quadric (or K-quadric) and the 5-dimensional space the Klein- (or K-) space. This viewpoint of three-dimensional line geometry has been used by authors before, but never as extensively as in this text. Both the classical material on the subject and new contributions by the author have been included.

The text is necessarily quite long and the author has tried to overcome some of the difficulties of length by dividing his work into "books," each of which can be read without the others. This naturally leads to some repetition of material. The books are divided into chapters which are numbered consecutively throughout the text. There are five books: the first (Chapter I) an introduction to line space; the second (Chapter II) on ruled surfaces; the third (Chapters III, IV, and V) on congruences; the fourth (Chapters VI–IX) on complexes, and the last (Chapter X) on line-space. Tensor calculus is used at all times, and simplifies the notation. For those readers who are unfamiliar with the tensor calculus the author has included in an appendix a straightforward, well written account of that part of the subject necessary for a reading of the text.

In the first book the author defines Plücker coordinates, Klein points, and Klein 5-dimensional space. He states that all topics, as