has the upper-bound-property. An appendix makes brief mention of a few related matters (such as infinite cardinals and the definition of a group).

The book contains many striking phrases designed to eliminate some common sources of confusion. Here are two examples. (1) "It should be stated emphatically that there is nothing undesirable about parentheses. Our efforts at removing them are directed toward devising a test to determine if two given expressions are equal." (2) "The reader is urged to distinguish between his everyday use of the words [*greater* and *less*] and the mathematical one; otherwise he will not be impressed by their similarities nor take seriously their differences."

There are also examples of bad didactic technique. They are all of the same type: some difficult concepts are not sufficiently motivated. Thus the empty set is introduced without so much as a by-your-leave and the possibility of a set being a member of itself is casually (and needlessly) referred to. The construction of the integers via ordered pairs is introduced by the following sentence: "The reader who finds our definitions arbitrary and somewhat bizarre should be informed that they are the product of a long evolution of which our exposition gives only the final stage." (The patronizing tone of this sentence recurs frequently.) The fact that the author is thinking of (7, 11) as $7-11$ is kept a secret from the reader.

Mathematically, the author's treatment of statements involving variables ("if x is even, then $x+1$ is odd") is probably unobjectionable but certainly peculiar. The assertion that "we shall only accept those statements that are definitions and those statements that can be proved to be logical consequences of the definitions" is somewhat startling, to say the least.

Enough has been said to communicate the flavor of the work. The book can be useful to a beginner as an outline of territory whose detail maps are not available to him. As such, the reviewer recommends it, but he recommends also that an experienced guide be taken along on the tour.

PAUL R. HALMOS

Theory of games and statistical decisions. By D. Blackwell and M. A. Girshick. New York, Wiley, 1954. 12+355 pp. $7.50.

Statistical decision theory originated in Wald's 1939 paper (Ann. Math. Statist. vol. 10, pp. 299–326), whose interest is now almost purely historical. It was designed to embrace all problems of statistical inference which are the raison d'être of statistics; with inessential modifications it still does. In its simpler form the statistician has to construct a (decision) function $\delta(x)$ from Euclidean n-space $X=\{x\}$ to a decision space $D=\{d\}$. Each point x represents a possible value of a chance variable (a sample) whose distribution function F is unknown to the statistician but is assumed to be one of a class Ω (which contains at least two and often infinitely many members). The "goodness" of δ for F is measured by a functional $r(F, \delta)$ of δ and F; in practice $r(F, \delta)$ is usually $\int W(F, \delta(x))dF(x)$, where W is a functional of F and $d=\delta(x)\in D$. A partial ordering among the δ is defined by: δ_1 "as good as" δ_2 if $r(F, \delta_1)\leqq r(F, \delta_2)$ for all $F\in\Omega$, and δ_1 "better than" δ_2 if also the inequality holds for at least one F in Ω. A class of decision functions is called complete if, for any decision function outside the class, there exists a better one in the class. A class of decision functions is called essentially complete if, for any decision function outside the class, there exists one in the class which is as good. In actual practice it is usually necessary to choose an "optimal" element or elements from an essentially complete class; for example, according to the "minimax principle" δ^* is optimal if

$$\sup_F r(F, \delta^*) = \inf_\delta \sup_F r(F, \delta)$$

and there exists no δ better than δ^*. In application to physical phenomena the physical scientist will want to use an optimal decision function, say δ^*, and will make decision $\delta^*(x)$ when x is the complex of his observations (his sample). In the general Wald theory the mathematical problems are enormously complicated by allowing the observations to be obtained sequentially, by having the theory indicate the choice of chance variables to be observed (design of experiments), and by permitting randomized decision functions.

The value of Wald's ideas for statistical theory can hardly be overestimated. They introduced a unifying clarity into the entire mathematical theory. A clear framework for the formulation of old and new problems was provided and the stimulus was given for a whole host of new ones. Decision theory, if understood by the practical statistician, would serve him in good stead in obtaining working solutions even for problems for which the optimal theory has not yet been developed.

We shall discuss below the theorems of decision theory on the relation between complete classes and Bayes solutions, and on the behavior of Bayes solutions; δ_ξ is a Bayes solution for the (a priori) distribution ξ on the space of F's if $\int r(F, \delta_\xi)d\xi=\min_\delta \int r(F, \delta)d\xi$. The other principal results, which should also be of interest to mathematicians in general, are the theorems on the elimination of randomization (Ann. Math. Statist. (1951) pp. 1–21 and 112–114) which gave rise to, and arose from, generalizations of Liapounoff's theorem in measure theory (Pacific Journal of Mathematics (1951) pp. 59–74, and Proc. Amer. Math. Soc. (1951) pp. 390–395), and the solution of an outstanding problem in sequential analysis, the proof of the optimum character of the Wald sequential probability ratio test (A: Ann. Math. Statist. (1948) pp. 326–339).

Wald's 1950 book (*Statistical decision functions*) contains five chapters. Chapter I gives the general formulation of the problems of statistics in terms of decision theory. Chapter II is on the theory of games. Chapter III is the heart of the book and contains theorems on complete classes, Bayes solutions, and minimax solutions. Most of the difficulties, many of them of measure-theoretic character, are caused by the general formulation of the problem. The proofs testify to Wald's great mathematical powers and ingenuity, but they are not of a polished and final sort. In fact (see discussion below) it is almost certain that in final form the approach will be different. Chapter IV is essentially the paper in Proc. Nat. Acad. Sci. U.S.A. (1949) pp. 99–102, hereafter cited as B, and the paper in Ann. Math. Statist. (1950) pp. 82–99. The final Chapter V contains complete classes, and more rarely, also minimax solutions, for problems designed to illustrate the theory. While most of these problems are interesting and ingenious, relatively few are important or vital. Thus a serious question could be raised whether the state of the theory was such as to justify the publication of Wald's book, not only because of the state of Chapter III, but also because of the paucity of vital practical applications of the theory.

The present book by Blackwell and Girshick eliminates the heart of the difficulties of Wald's Chapter III by a drastic reduction in the mathematical generality, in part by a restriction to discrete distributions. It contains elaborate discussions of the formulation of the decision problem, and several chapters on the theory of games. Chapter IV of Wald's book is essentially reproduced, and there are ad hoc examples of the type of Wald's Chapter V. In addition there is some standard statistical material on sequential analysis, sufficiency, and point estimation, which is not usually classified under decision theory. The book concludes with results of one of the authors on comparison of experiments in the case of discrete distributions and finitely many alternatives.

Whatever doubts may be raised about the justification for Wald's book apply a fortiori to the present book, published four years after Wald's. Very few essential results not included in Wald's book are to be found in the present book. The practical statistician will not find in either book answers to the problems which confront him. He would be introduced to an approach of immense value to him, but it is obvious that Wald's book is mathematically inaccessible to most practical statisticians, and this is so, in the opinion of the reviewer, with the present book. It might be expected then that the present book would appeal to mathematicians, but the reviewer has misgivings on this score. The book is intended primarily as a textbook for first year graduate students in statistics. It is very formal in presentation; there are numerous definitions, very formally stated. The reviewer counted twenty-four rolling definitions in Chapter 3 and only eight theorems, many of the latter of very modest content. A boring effect may be created among mathematicians by the practice of elevating minor and obvious arguments to the level of theorems and lemmas, with all that this implies in notation and in obscuring the main results; for example, Lemma 8.4.1 is a typical example of a most common mathematical argument which would ordinarily rate a casual sentence. Thus, the presentation is too imposingly formal for practical statisticians, and may be too boringly formal and the mathematical content may be too dilute to be interesting to mathematicians.

One of the achievements of the Wald theory is the proof of the optimum character of the Wald sequential probability ratio test. In the original proof given in A, the first two lemmas yield a convexity result which is more expeditiously given in B. The proof in the present book follows the lines of that of A with the improvements from B just noted. In their proof (page 292) the present authors appear to effect an economy in the argument by omitting the arguments of §§5 and 6 of A. The purpose of these sections was to enable one to conclude that, for any given a priori distribution, there exist non-negative weights w_{12} and w_{21} (Blackwell and Girshick's notation) which will make any given sequential probability ratio test a Bayes solution. The present authors argue that this conclusion follows from the fact that equations (10.3.7) and (10.3.8) are linear in w_{12} and w_{21} and so can be solved for w_{12} and w_{21}. However, the crucial question is whether such w_{12} and w_{21} exist. Equations (10.3.7) and (10.3.8) are not homogeneous in w_{12} and w_{21}, and the authors have not proved that these equations have a solution (are always consistent). Not only this, but the authors have failed to prove that there is a non-negative solution, which is essential for the argument. Thus the authors have achieved an "economy" of argument at the expense of an essential gap in the proof; this gap is present also in their paper in Econometrica (1949) pp. 213–244. (A minor detail not related to the above:

The proof is also not valid unless one postulates, as is done in A, that $E(n|S; H_1)$ and $E(n|S; H_2)$ are both finite.)

The authors' view is that decision theory is the study of what they call statistical games, and a branch of the general theory of games. The essence of a zero-sum game (which is obviously the kind of game meant) is the conflict caused by the opposing interests of the players. What makes (von Neumann's) main theorem for the zero-sum two-person game so pretty is its ingenious resolution of this conflict. What is a reasonable resolution of the conflict in the case of the n-person game is still under debate. It is misleading to think of statistical theory as a zero-sum two-person game. The conflict which is the essence of the game is not present (i.e., there is no basis or need for regarding Nature as a malevolent opponent of the statistician), and it is not especially useful to introduce the terminology of game theory in obtaining results in decision theory. Mathematically speaking, this means that decision theory is not a part of game theory, and that a book on decision theory should not make decision theory a branch of game theory. But, it will be remonstrated, is not Wald's book based on this game-theoretic approach? The answer is that, at the time Wald wrote his book, it was not known that Bayes solutions correspond simply to supporting planes for the set of risk points in the space of the F's. Not knowing this Wald used the equation

$$(*) \qquad \inf_{\delta} \sup_{\xi} r(\xi, \delta) = \sup_{\xi} \inf_{\delta} r(\xi, \delta)$$

as his principal tool for proving complete class theorems, and consequently found it indispensable to set up a structure in which he could prove this equation. In one of the very few statistical papers written after the completion of his book and before his untimely death in 1950 (Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950, pp. 149–157) the true situation is understood, the formulation and the arguments involve no game theory whatever, and the complete class theorem is derived by a supporting plane argument. It is desirable, in an exposition of decision theory, to show the connection with game theory, namely that, when (*) holds, it is sometimes easier or suggestive to find a minimax δ by first finding a ξ least favorable to the statistician. It may also be considered desirable to enlarge the student's education by inclusion of a section on game theory in a textbook on decision theory. But, in the light of our present knowledge, it is misleading to consider decision theory a special branch of game theory.

The book contains a very lengthy bibliography in which the relatively few essential papers on decision theory are submerged. The theorems are numbered serially without any indication whatever of their authorship. When one considers that the entire subject is only fifteen years old, that the majority of present results are due to Wald, and that the number of other significant contributors to the theory is very small, it would seem that a due regard for history would be neither difficult nor amiss. The authors cite in the bibliography their own unpublished work on invariant procedures, but have neglected to cite earlier unpublished work by M. P. Peisakoff (*Transformation of parameters*, Princeton Thesis, 1951).

The brief chapter on utility and principles of choice is a desirable part of the book. The subject certainly deserves discussion, and more than the casual remarks accorded to it in Wald's book. The authors' discussion summarizes most of the meager results on the subject.

There are many exercises, some of which should be instructive to the student. In this respect the present book is much superior to Wald's, which contains no exercises.

The number of definitions and the amount of notation are so great that an indexed glossary would be of great help in reading the book.

We close with a word about further problems. Some work is now being done which aims at an elegant reworking of essentially Wald's Chapter III. The natural method is to try to choose a proper topologic space in which support planes will represent Bayes solutions. While there is still a certain amount of this left to do and this work has a value, it is not likely to constitute a deep advance, and there are definite limits to its mathematical interest. It seems to the reviewer that essential advances remain to be made in the direction of more detailed behavior of general Bayes solutions and of detailed description of Bayes solutions for important special problems. Methods are needed for obtaining optimal (e.g., minimax) solutions. The problem of choice and, in particular, consequences of proposed solutions to the problem of choice, require much investigation. The theory of decision functions can undoubtedly be employed to illuminate and deepen results in other branches of statistics; recent application to the maximum likelihood method is a case in point. In any case there remain many problems which are interesting and difficult from the mathematical point of view and which show that the subject has great mathematical vitality. Of its practical statistical importance there can be no doubt.

Added in proof. Dr. M. A. Girshick died in March, 1955. He was Professor of Statistics and leader of the statistics group at Stanford

University. His untimely death removes from the scene one of the early American workers in statistics.

J. WOLFOWITZ

Die Grundlagen der Quantenmechanik. By G. Ludwig. Berlin, Springer, 1954. 12+460 pp., 52 figures. Paperbound, 49 DM; clothbound, 52.60 DM.

This book gives a clear and interesting account of the quantum mechanics of a non-relativistic system with a finite number of degrees of freedom. This is a subject whose main lines, particularly consistency and uniqueness, are relatively well established. Roughly the first half of the book is devoted to a careful laying of the foundations of the subject from both the mathematical and physical sides. The second half develops the crucial applications and techniques in a thoroughgoing and readable way. The material of the book is well integrated and evenly presented, there being a good balance between the demands of intelligibility and those of brevity, with most doubtful situations being resolved in favor of the former. Though much of the mathematical and some of the physical material is essentially contained in von Neumann's well known book, the present work by its larger size and more limited scope provides a more spacious and elementary exposition. While its emphasis is on the physical side, its attitude is primarily logical, so that the mathematics is treated as a basic part of the scheme, rather than as a necessary evil. On the whole the book is solid and spirited, eclectic rather than pure, and should be unusually useful for some time to come.

The book includes somewhat condensed but readable treatments of Hilbert space and of group representation theory, in so far as they are used by it. Elsewhere in the book the mathematics makes use of assumptions that are generally clearly formulated so that the whole work is essentially quite rigorous. Important developments involving a relatively high level of mathematical sophistication are however omitted, among them von Neumann's theorem on the uniqueness of the Schrödinger operators and Kato's recent work in the Transactions on n-particle systems.

It is a significant anomaly that there is no material in this treatment of foundations on either quantum fields or relativistic invariance. The foundations that are presented are roughly as valid, from a physico-mathematical viewpoint, as the usual foundations of the Newtonian dynamics of a finite set of particles. Either theory is logically consistent as well as mathematically categorical and self-