papers by K. Menger[1] concerned with the validity of the law of diminishing returns. This is: *Let the product of applying y* (e.g., *dollars*) *to x* (e.g., *acres*) *be $E(x, y)$. Then for $h > 0$*

$$E(x, y_2 + h) - E(x, y_2) < E(x, y_1 + h) - E(x, y_1)$$

provided $y_2 > y_1 > y = y(x)$. Various properties of production functions E such as monotony, super-additivity, super-homogeneity, dependence are introduced. Examples are presented to show how these are related among themselves and with the above law and similar propositions. These relations are summarized graphically. This essay can be recommended by mathematicians to their friends in social science as an example of how one discipline can be used in another.

The concluding essay by Morgenstern is concerned with the role of experiment and computing in economics. Apart from discussions of a more philosophical nature there are indications of some computational experiments of immediate interest to economists.

OLGA TAUSSKY

Mathematics and plausible reasoning. By G. Pólya. Volume I, *Induction and analogy in mathematics*, 16+280 pp., $5.50; volume II, *Patterns of plausible inference*, 10+190 pp., $4.50. Princeton University Press, 1954. 2 vols., $9.00.

There are many so-called "popular" books on mathematics. Some of them turn out to be of interest to professional mathematicians only (or, perhaps, to professional mathematicians *in ovo* as well). Others are so non-technical as to be well within the reach of any educated layman, and, consequently, their subject hardly deserves to be called mathematics. Most of the time Pólya manages to steer an admirable course between these two extremes. The two volumes under review are, however, not uniform in this respect; the first is more the mathematician's volume and the second the philosopher's. Since this review is addressed to mathematicians, it will discuss the first volume in more detail, and, it may well be charged, with more sympathy, than the second.

The book as a whole is organized around the central thesis that a good guess is quite as important as a good proof. As in his little book *How to solve it*, Pólya advocates that the mathematician should think and talk (at his desk and in the class room) about the theory of guesses as well as the theory of proofs. "Certainly, let us learn proving," he

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[1] *Bemerkungen zu den Ertragsgesetzen* and *Weitere Bemerkungen zu den Ertragsgesetzen* in Zeitschrift f. Nationalökonomie vol. 7 (1936) pp. 25–26 and 388–397.

says, "but also *let us learn guessing*." The function of the first volume in the theory of plausible reasoning is to provide some concrete mathematical raw material; the second volume is more interested in an abstract philosophical discussion of the patterns that the first volume indicates.

More than half of the first volume is occupied by problems (at the end of each chapter) and their solutions (at the back of the book). The organization of these problems is similar to that of the well known problem collection by Pólya and Szegö. Sometimes the very statement and sometimes the solution of one problem suggests the next problem; by working through a graded series of problems of this sort the reader gets a very good insight into their subject. The problems are almost always non-trivial. The first problem in the first chapter is to guess the rule according to which the successive terms of the following sequence are chosen: 11, 31, 41, 61, 71, 101, 131, · · · · . (For the convenience of the reader of this review the solution is given below.) The last problem in the last chapter concerns some computations involving the torsional rigidity of a beam with square cross-section. Many of the problems invite the reader to guess an answer, to examine a situation, or to conjecture a theorem.

The problems, by the way, are not called problems; they are called examples and comments. Roughly speaking, the examples are mathematical exercises and the comments are philosophical discourse about them. The quality of the comments is quite variable. Some are hardly more than weak jokes (there is one about the differences among logicians, mathematicians, physicists, and engineers), while others are penetrating remarks on mathematical methodology (e.g., a discussion of the effect of strengthening the hypothesis of a theorem to be proved by induction).

The greater portion of the body of volume I (i.e., of the material distinct from the problems and their solutions) is mathematical exposition of unusually high caliber; only about 40 pages (sprinkled through the volume) are devoted to outright philosophizing. An idea of the contents can be gained from the following list of the main mathematical ideas that occur in the various chapters. (The titles of the chapters would be of little use here; Chapter VI, for instance, is entitled *A more general statement*.) I. Goldbach's conjecture. II. The Pythagorean theorem; Euler's method of evaluating $\sum_{n=1}^{\infty} n^{-2}$. III. The Euler formula, $V - E + F = 2$. IV. Sums of squares; in particular, the four-square theorem. V. If $a_n > 0$ $(n = 1, 2, 3, \cdot \cdot \cdot)$, then

$$\lim \sup_n ((a_1 + a_{n+1})/a_n)^n \geqq e.$$

VI. Euler's recursion formula for the sum of the divisors of n. VII. Mathematical induction. Some non-routine applications, such as the following one: "If the polygon P is convex and contained in the polygon Q, the perimeter of P is shorter than the perimeter of Q." VIII. Maxima and minima, with, again, several ingenious, non-routine applications. IX. Minimum principles from optics and mechanics. X. The isoperimetric problem. XI. Miscellaneous problems. Sample: if the intersection of a solid sphere with a solid cylinder whose axis passes through the center is removed from the sphere, find the volume of the remainder in terms of the radius of the sphere and the height of the cylindrical hole.

The second volume is quite different from the first. The problems occupy less space (less than a third of the volume) and play a less important role. Their character is also different. Sample: "Check Heron's formula [for the area of a triangle in terms of the sides] as many ways as you can." Chapter XII discusses the "fundamental inductive pattern" and some of its variations. (If A implies B, then the discovery that B is true makes A more credible.) There is an interesting discussion of the author's teaching methods, centered around the formula for the area of the lateral surface of the frustum of a right circular cone. Chapter XIII continues in the same vein. By way of an example there is a long discussion of judicial proof with many details of a murder case. The best part of the volume for the mathematician is about a dozen pages (in the examples and comments on Chapter XIII) that discuss some of Pólya's own work with historical and psychological side lights. Chapters XIV and XV concern probability, mostly from a non-mathematical point of view. The main emphasis of the last chapter, Chapter XVI, is on the applications of the preceding discussion to pedagogic methods.

Neither volume has an index; there is, instead, a very detailed analytical table of contents. The physical appearance of the volumes is excellent. The style throughout is informal and charming.

<div align="right">PAUL R. HALMOS</div>

P.S. The sequence 11, 31, 41, 61, 71, 101, 131, $\cdots$ consists of the primes ending in 1.

Elements of algebra. By Howard Levi. New York, Chelsea, 1954. 160 pp. $3.25.

This book is addressed to beginning students of mathematics; it is, roughly speaking, a text on freshman algebra. It is a text on freshman algebra in the sense that it talks about the removal of parentheses and