ble! Chapter V takes up Carathéodory's theory of abstract "exterior measures," i.e., set functions containing as particular cases the exterior measures deduced from a measure; these are axiomatically characterized among all "general exterior measures," and a similar characterization is given for interior measures. The chapter also includes the Carathéodory criterion for measurability of closed sets in a metric space; finally the connection is made between this theory and that developed in chapter I. The last chapter treats measure theory in boolean algebras and σ-complete boolean algebras. An appendix gives the "transfinite" generation of Borel sets and their main properties in finite-dimensional spaces.

The author's claim that he has "reached or even gone beyond" the limit of what is known today on the subject can hardly be accepted without restriction; for instance, no mention is made, in the last chapter, of the Stone and Loomis representation theorems, although they make the developments of that chapter practically pointless! No mention is ever made of characteristic functions of sets, which would at times make proofs much easier (for instance, the well known derivation of the "boolean ring" structure of a boolean algebra). Finally, the reviewer wants to take exception to the author's statement that measure theory (as understood in this book) is the foundation of the theory of integration. This *was* undoubtedly true some years ago, but is fortunately no longer so, as more and more mathematicians are shifting to the "functional approach" to integration. It is always rash to make predictions, but the reviewer cannot help thinking that, despite its intrinsic merits, this book, as well as its brethren of the same tendency, will in a few years have joined many an other obsolete theory on the shelves of the Old Curiosity Shop of mathematics.

J. DIEUDONNÉ

Lectures in abstract algebra. Vol. II. *Linear algebra.* By N. Jacobson. New York, Van Nostrand, 1953. 12+280 pp. $5.85.

Linear algebra is now universally recognized as perhaps the most important tool of the modern mathematician; its concepts and methods, moreover, when properly reduced to their essential features, are among the simplest and most straightforward imaginable. Nevertheless, it is still not uncommon to find graduate students who are totally unfamiliar with some of the fundamental notions of linear algebra, such as, for instance, the theory of duality. This may perhaps be attributed to the scarcity of good textbooks on the subject; if so, the present volume will undoubtedly do much to remedy this situation. Although this is the second part of a work which will

ultimately be a treatise covering the whole of what may be called "elementary modern algebra," the author has taken great pains to make the book as self-contained and as easy to read as possible. Indeed, pedagogical intentions are apparent throughout: for instance, definitions which are particular cases of others already given in the first volume are often stated again, sometimes twice or more, and with increasing generality. In addition, the text is accompanied by many well chosen examples and exercises, which provide ample opportunities for the student to test his understanding of the theory.

The concept of finite-dimensional vector space over a division ring is introduced in the first chapter, first for the example of n-tuples, and then more abstractly. The fundamental theorems on linear dependence, bases and dimension are proved; matrices (defined in vol. I) are connected with sets of vectors considered in relation with a given basis, and this connection is immediately used to link properties of matrices with linear dependence. The chapter ends with a study of the lattice of subspaces of a finite-dimensional space, and in particular of the notion of (finite) direct sum.

Chapter II is devoted to the general theory of linear transformations between finite-dimensional vector spaces. Matrices are now connected with linear transformations, as well as with systems of linear equations. The second part of the chapter takes up the theory of duality, following rather closely the corresponding treatment in the books of Halmos and Bourbaki on linear algebra.

Chapter III is almost as long as the first two chapters together, and develops the theory of elementary divisors of a linear transformation in a vector space over a field. From the pedagogical motives mentioned above, the author does not begin by giving the theory of finitely generated modules over a principal ideal ring; this only comes in the middle of the chapter, after the link between the two theories has been made perfectly natural. The theory of invariant factors is then used to complete the reduction of a square matrix over a field; the usual results are proved, in particular the Hamilton-Cayley theorem; at the end of the chapter the centralizer of a matrix is entirely determined, and related to the notion of a ring of endomorphisms of a module.

A shorter chapter follows, on sets of linear transformations, where the notions of invariant subspace, decomposability, reducibility and complete reducibility are discussed, and illustrated in particular for a set consisting of a single transformation, thus linking this chapter with the preceding one; the common reduction of a commutative set of linear transformations is given at the end of the chapter.

The next three chapters are concerned with bilinear forms. They

are defined in the general case of two vector spaces, one left and one right, over the same division ring, and their relation to linear transformations and duality theory is duly emphasized. The general theory is then specialized to a bilinear form over the product of a left vector space by itself, when the division ring possesses an anti-automorphism; this leads to the theory of hermitian and alternate bilinear forms, whose reduction is treated in detail. The final part of the chapter contains a proof of Witt's theorem (the first to appear in a textbook) in its full generality (including the case of characteristic 2 when every element of the division ring is a "trace" $\xi + \bar{\xi}$); some fragmentary results are also given on symmetric non-alternate forms over a field of characteristic 2.

The classical theory of euclidean and unitary spaces (of finite dimension) over the real field is developed in chapter VI, including the usual reduction and factorization theorems for normal, symmetric, hermitian, orthogonal, and unitary matrices. A short paragraph on analytic functions of matrices yields the exponential relation between hermitian and unitary transformations.

The next chapter is much more abstract, and gives the basic facts on tensor products of vector spaces and of linear transformations. The notion of tensor product is introduced in connection with the theory of duality, the tensor product $E' \otimes F$ of the dual E' of a left vector space E and of a left vector space F being in a natural correspondence with the group $\mathcal{L}(E, F)$ of all linear mappings from E to F. It is shown how for two-sided vector spaces the tensor product can again be made into a two-sided vector space; the rest of the chapter is concerned with the most important case, in which the vector spaces are over a field. There is a short description of the principal algebraic features of tensors and some of their symmetry classes; also included in this chapter are the extension of the field of scalars of a vector space and the definition of the tensor product of two algebras.

The last two chapters develop the beginning of the theory of rings of linear transformations. The finite-dimensional case is treated in chapter VIII, which in a very short space gives a neat description of the fundamental relationship between subspaces and ideals, as well as the characterization of automorphisms of the ring of linear transformations, all this being done without writing down a single matrix! Finally, chapter IX extends the previous notions and results to infinite-dimensional vector spaces; it is definitely on a higher level than the rest of the book, and is probably its most attractive feature, because of the variety of new topics which are here again presented in book form for the first time, and the elegance of their pres-

entation; as an introduction to the modern theory of rings (which will be treated much more exhaustively in a forthcoming book of the author), it should make fascinating reading for any advanced student. Among the topics treated are dimensionality in infinite-dimensional vector spaces, including a determination of the dimension of the dual space; the "finite" topology on spaces of linear transformations, with the notion of total subspaces and of dense rings; the determination of two-sided ideals and of isomorphisms of certain rings of linear transformations; a discussion of dense rings of linear transformations possessing involutorial anti-automorphisms; and finally the author's well known density theorem, with classical applications to irreducible algebras of linear transformations.

The choice of notations and terminology might at times, in the reviewer's opinion, be definitely better. For instance, the author sticks to his habit of writing linear transformations on the right of the variable; there is no objection against this as long as one remains in pure algebra; but linear algebra is now the daily bread of mathematicians from every part of the horizon, and its results should be immediately available to them, without having to use mirrors to restore the operator to what is considered by the majority of analysts as its proper place! It is amusing, by the way, to see how the author himself violates his own convention when it comes to writing linear forms, for instance! The term of "direct product" and the notations $x \times y$ and $E \times F$ for the tensor product are quite unfortunate, since they already have several other meanings; why not adopt the von Neumann symbol $\otimes$, now almost universally used? Writing $\langle x, f \rangle$ or $\langle f, x \rangle$ for $f(x)$ in duality theory would clarify many a formula and emphasize the symmetry of the results. The fundamental notion of characteristic vector of a matrix is relegated to an exercise; so is the relation dim $(E+F)$ + dim $(E \cap F)$ = dim E + dim F, although a whole page is devoted to the "modular" law in the lattice of subspaces, which is never used any more (and is there apparently for the sake of those who still believe lattice theory is an important part of mathematics!).

But these are very minor criticisms of a remarkable book, which has every chance of becoming a classic in its field for many years to come.

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The Michigan Mathematical Journal. Vol. 1. University of Michigan Press, Ann Arbor, 1952. 197 pp. $2.00 to individuals ordering directly. $4.00 to others.