theorems and formulae for reference. The book has the usual excellent typography of the Cambridge tracts and is almost free of misprints.

LOWELL SCHOENFELD

Sur les espaces fibrés et les variétés feuilletées. By W. T. Wu and G. Reeb. (Actualités Scientifiques et Industrielles, no. 1183.) Paris, Hermann, 1952. 157 pp.

This monograph consists of the theses of the authors at the University of Strasbourg under the direction of C. Ehresmann, and contains detailed accounts of original contributions of the authors, whose main results have been announced in the Comptes Rendus de l'Académie des Sciences à Paris. The contents of the two papers are not directly related, although both can be said to be concerned with certain aspects of the theory of differentiable manifolds.

The paper of Wu Wen-Tsun has the complete title: *Sur les classes caractéristiques des structures fibrées sphériques.* Its starting-point is the so-called universal bundle theorem. In the cases in which the author is interested, the fiber bundle has as base space a finite polyhedron B and as structural group G one of the three groups: the orthogonal group O_m in m variables, the proper orthogonal group $\hat{O}_m$ in m variables, and the unitary group O'_m in m complex variables. The universal bundle theorem asserts that, B and G being given, there exists a bundle with the base space B_0 and the same structural group G such that the given bundle is induced by a mapping $f:B \rightarrow B_0$ and that this mapping f is defined up to a homotopy. It was perhaps Pontrjagin who first observed that the dual homomorphism $f^*:H(B_0, R) \rightarrow H(B, R)$ of the cohomology ring of B_0 into the cohomology ring of B, relative to a coefficient ring R, is thus completely determined by the bundle. In our three cases we can take as B_0 respectively the following Grassmann manifolds: the manifold $R_{n,m}$ of all m-dimensional linear spaces through a point O in a real Euclidean space of dimension $n+m$, the manifold $\widehat{R}_{n,m}$ of such oriented linear spaces, and the manifold $C_{n,m}$ of all linear spaces of dimension m through a point O in a complex Euclidean space of complex dimension $n+m$, with n sufficiently large in all three cases. The author's notation for $C_{n,m}$ is slightly confusing. In many places, such as on pages 7, 8, 48, etc., it would be clearer to write $C_{n',m'}$ for $C_{n,m}$.

The fruitfulness of this approach is based on the fact that these Grassmann manifolds are relatively "rich" in homology properties. The homology groups of these manifolds were determined by Ehresmann, on adopting the notion of Schubert varieties in algebraic geometry. Since f^* is a multiplicative homomorphism, it suffices to know its effect on a multiplicative base of $H(B_0, R)$. Moreover, we can restrict ourselves to the cohomology classes of dimensions $\leqq \dim B$. The cases of interest are the following:

(1) The structural group G is O_m or $\hat{O}_m$ and the coefficient ring R is the finite field of two elements. In this case a multiplicative base of $H(B_0, R)$ is formed by the k-dimensional cohomology classes W_2^k, $1 \leqq k \leqq m$. Their images under f^* are called the Stiefel-Whitney classes.

(2) G is O_m or $\hat{O}_m$ and R is the field of real numbers. There are classes P_0^{4k}, $1 \leqq k \leqq m/4$, which generate the elements of dimension $\leqq m$ of $H(B_0, R)$. Their images under f^* are called the Pontrjagin classes.

(3) G is O'_m and R is the ring of integers. There are classes C^{2k}, $1 \leqq k \leqq m$, of dimension $2k$ which generate the elements of dimension $\leqq 2m$ of $H(B_0, R)$.

These cohomology classes are what the author calls characteristic classes. It is the study of their properties and their geometrical applications which constitutes the main aim of this paper.

The paper is divided into five chapters. Chapter I contains an exposition of the homology properties of Grassmann manifolds. The results are essentially due to Ehresmann, with modifications by Pontrjagin for the case $\widehat{R}_{n,m}$. The section on the relation between two dual Grassmann manifolds is new and should be found useful in geometrical applications.

The author considers Chapter II as containing his main contribution in this paper. For simplicity call a fiber bundle with the structural group G a G-structure. If G' is a subgroup of G, a fundamental problem in fiber bundles is whether a G-structure is equivalent to a G'-structure. The author studies the cases when (G, G') are respectively the pairs $(O_m, \hat{O}_m)$ and $(\hat{O}_{2m}, O'_m)$. For definiteness of description let us restrict ourselves to the pair $(\hat{O}_{2m}, O'_m)$. There is a canonical mapping $\phi'\colon C_{n,m} \to \widehat{R}_{2n,2m}$. If an $\hat{O}_{2m}$-structure is induced by a mapping $f\colon B \to \widehat{R}_{2n,2m}$, it follows from the universal bundle theorem that it is equivalent to an O'_m-structure if and only if there exists a mapping $g\colon B \to C_{n,m}$, such that f and $\phi' \circ g$ are homotopic. The study of the homology properties of the mapping ϕ' is thus carried out. From this one derives the following necessary conditions for a given $\hat{O}_{2m}$-structure to be equivalent to an O'_m-structure

$$(1) \qquad W_2^{2k} = C_2^{2k}, \qquad W_2^{2k+1} = 0$$

$$(2) \qquad (-1)^k P_0^{4k} = \sum_{h=0}^{2k} (-1)^h C_0^{2h} \cup C_0^{4k-2h},$$

where C^{2h} are the characteristic classes (with integer coefficients) of the O'_m-structure and the subscripts 2 and 0 denote respectively their reductions to coefficients mod 2 and to real coefficients. Neither set of these conditions is trivial, even for the tangent bundle of an even-dimensional orientable manifold. In fact, the author gives examples of even-dimensional orientable manifolds whose three-dimensional Stiefel-Whitney class does not vanish. (Whitney proved that the three-dimensional Stiefel-Whitney class of a four-dimensional orientable manifold always vanishes.) Meanwhile, the author derives from (2) the theorem that a sphere of dimension $4k$ has no almost complex structure.

Chapter III is devoted to the study of the homotopy properties of Grassmann manifolds. The main result is a homotopy classification of the mappings of a four-dimensional complex into $C_{n,2}$, $n \geqq 2$. This result is applied in Chapter IV to derive a necessary and sufficient condition for a four-dimensional oriented manifold to have an almost complex structure. The author even determines all possible almost complex structures on it. (In the statement of Theorem 10, p. 74, it would be clearer to mention that the manifold M is of dimension 4.) The same idea is applied to derive a necessary and sufficient condition that a six-dimensional oriented manifold has an almost complex structure.

From an $\hat{O}_m$-structure and an $\hat{O}_n$-structure over the same base space we can define, following an idea of Whitney, an $\hat{O}_{m+n}$-structure. The relations between the characteristic classes of these three structures constitute the duality theorems of the Whitney type. Chapter V gives the statements and proofs of such duality theorems. Whitney himself has not published his proofs, claiming that they are very difficult. The present method of looking at the problem as the study of the homology properties of a certain canonical mapping of Grassmann manifolds seems to be the proper approach to this question.

This thesis was finished in the summer of 1949. In the next two years Wu pushed this study much further. Among other results he has determined the Steenrod squares on a Grassmann manifold and has obtained necessary conditions that a sphere bundle be the tangent bundle of a differentiable manifold. (Cf. Comptes Rendus Paris, v. 230, pp. 508–511, 918–920.)

The second paper of this book, by G. Reeb, is entitled: *Sur certaines propriétés topologiques des variétés feuilletées*. Its purpose is the study of the global properties of the integral manifolds of a completely integrable Pfaffian system on a manifold. The given data consist of a differentiable manifold V_n of dimension n and class 2 and on it a field E_q of elements of contact of dimension q, $1 \leqq q \leqq n-1$, and class one, which is completely integrable. The latter means that every point $x \in V_n$ has a neighborhood Ω_x and a homeomorphism h_x of class one of Ω_x into an open subset of the Euclidean space R_n of n dimensions, such that the elements of contact of E_q go into parallel elements. If we suppose E_q to be spanned by q linearly independent vectors, each of class one in a certain neighborhood, it follows from a well known theorem on differential equations that the condition of complete integrability of the field can be expressed as differential equations in the components of the q vector fields. The unpleasant feature of the situation is that these differential equations are quadratic, and not linear. The question of the existence of a field E_q on V_n, not necessarily completely integrable, is a question on fiber bundles, equivalent to that of the existence of a cross section in a certain associated bundle. The author mentions as a fundamental problem the following: If a field E_q exists on V_n, does there exist one which is completely integrable?

This is undoubtedly an important problem in differential geometry. The author believes that the answer would be negative, a conjecture which the reviewer is willing to share. But no counter-example is known. On the contrary, on the three-sphere and on some other three-dimensional manifolds, the author gives examples of E_2, which are completely integrable (p. 112).

The reviewer wishes to point out that there are analogous situations in which the answer to such a question is negative. For the general problem can be described as that of deciding whether a fiber bundle has a cross section satisfying a system of differential equations, when a cross section does exist without satisfying the differential system. The simplest case is that of a parallelisable manifold in which the structural group of the tangent bundle can be reduced to the identity. For a manifold V_n to be parallelisable it is necessary and sufficient that there exist n linear differential forms ω^i, $1 \leqq i \leqq n$, of class one, which are everywhere linearly independent. Their exterior derivatives can be written

$$d\omega^i = \sum_{j,k=1}^{n} c^i_{jk}\omega^j \wedge \omega^k, \qquad i = 1, \cdots, n, \; c^i_{jk} + c^i_{kj} = 0,$$

where c^i_{jk} are functions in V_n and are completely determined. The question whether a parallelisable V_n has a set of ω^i satisfying the further condition that the c^i_{jk} are constants is a problem of the above type. When V_n satisfies a completeness condition, which it does if, for instance, its universal covering manifold is compact, this amounts to saying that V_n is a Lie group. This is not always true for a parallelisable manifold; for example, the seven-dimensional sphere is parallelisable but is not a Lie group. There are other questions of similar nature, but we shall not bother with them here.

The object of the author is to study the integral manifolds of the completely integrable field of elements E_q. Since the classical work of Poincaré on the integral curves of ordinary differential equations this problem has received much attention. It is well known that even for $q=1$ and even under analyticity assumptions the global behavior of the integral curves can be rather "wild." Complete information is known in the case of integral curves in the plane, owing to the work of Kaplan.

A natural step in such a study is to introduce a new topology in V_n, whose open sets are unions of integral manifolds. The manifold V_n with this topology is called a leaved manifold (variété feuilletée) and a connected component of it a leaf (feuille). Chapter A of this paper is devoted to the study of some of the basic properties of this topology. Its last section contains several examples of leaved manifolds, which should prove useful in any further study of the subject.

Chapter B studies the topological properties of neighboring leaves, the results of which are called theorems of stability. Three types of stability theorems are proved, the first concerned with the behavior of leaves in the neighborhood of a leaf, the second with the effect on the leaves when the leaved structure is slightly modified, and the third with the case in which $q=n-1$, where more precise information can be obtained. As examples of such results we mention the following: If a leaf is compact and has a finite fundamental group, every open neighborhood of it has a saturated open neighborhood such that every leaf which meets it is compact. In particular, if the leaved structure is of class 2 and if all the leaves are compact and simply connected, they are homeomorphic and define a fiber structure in V_n, whose base space is a manifold of dimension $n-q$. The third stability theorem implies as a corollary the following theorem: If $q=n-1$ and if V_n is compact and connected and has a compact leaf with a finite fundamental group, every leaf is compact and has a finite fundamental group.

Chapter C is concerned with the case $q=n-1$ and studies the isolated singularities of a completely integrable Pfaffian form. In a neighborhood with the coordinates x^i, $i=1, \cdots, n$, such a form can be approximated by a form $\omega' = \sum_{i,j=1}^n a_{ij}x^j dx^i$, where a_{ij} are constants. It is proved that the matrix (a_{ij}) is either symmetric or of rank 2. In the former case the singularity is called a center, if the form $\sum_{i,j=1}^{n} a_{ij}x^i x^j$ is positive definite. When V_n is compact, with $n \geqq 3$, a completely integrable field E_{n-1} with a nonempty set of centers as singularities must have leaves which are homeomorphic to an $(n-1)$-sphere. Moreover, this happens only when there are exactly two centers and when V_n is homeomorphic to an n-dimensional sphere. The chapter is concluded by a more detailed study of the case when the form is analytic.

There does not seem to be any doubt to the reviewer that both studies contain valuable contributions to the topology and differential geometry of manifolds. We also believe that they only mark a beginning of further fruitful investigations.

SHIING-SHEN CHERN

Grundzüge der theoretischen Logik. By D. Hilbert and W. Ackermann. 3d ed. Berlin, Springer, 1949. 8+155 pp.

Principles of mathematical logic. By D. Hilbert and W. Ackermann. Trans. by G. G. Leckie and F. Steinhardt; ed. and with notes by R. E. Luce. New York, Chelsea, 1950. 12+172 pp.

The first edition of this book appeared in 1928. According to the preface, it was based on Hilbert's lectures of 1917–22. It was reviewed, somewhat unsympathetically, by Langford in this Bulletin, Vol. 36, pp. 22 ff. The second edition appeared in 1938; it was reviewed by Rosser in this Bulletin, Vol. 44, p. 474, and by Quine in Journal of Symbolic Logic, Vol. 3, p. 83. The third edition and the English translation of the second edition, with both of which this review is concerned, appeared almost simultaneously in 1949–50. They have been previously reviewed in the Journal of Symbolic Logic, Vol. 15, p. 59, by Church, and Vol. 16, p. 52, by Zubieta, respectively.

The book was intended as an introductory textbook of mathematical logic in a narrow sense. The Hilbert school never subscribed to the identification of mathematics and logic, and regarded "mathematical logic," "theoretical logic," and "logical calculus" as synonymous designations for a preliminary stage in the subject of "foundations of mathematics," which many Americans prefer to call "mathematical logic" in a broader sense (cf. Quine's book of 1940). Anything depending on an axiom of infinity or similar assumption would belong to the latter subject but not to the former. Nevertheless, the authors regard the narrower subject as an essential step to the broader. Thus, in the first preface, signed by Hilbert, it is stated that the book is intended as a preparation for a further book by him and Bernays,