

probably meant is that the "double" integral $\iint f(x, y) d\nu(x, y)$, where ν is the product measure of, say, λ and μ , is alternatively denoted by $\iint f(x, y) d\lambda(x) d\mu(y)$; the corresponding iterated integrals are denoted by symbols such as $\int d\lambda(x) \int f(x, y) d\mu(y)$. The threat to use two integral signs with only one differential is never actually carried out. Aside from this triviality, I noticed no errors in the book, trivial or otherwise. I caught only three minor misprints; the only one that might worry a student for a few seconds is on p. 121, line 9: f_n should be \tilde{f}_n .

The topics centering around the names of Fubini (decomposition of measures), Radon and Nikodym (absolute continuity), and Haar (group invariance), are not discussed in this volume; according to the introduction, they will be treated in the subsequent chapters.³ The introduction indicates also that the authors are planning eventually to apply their theory (probability) and to generalize it (distributions).

My conclusion on the evidence so far at hand is that the authors have performed a tremendous *tour de force*; I am inclined to doubt whether their point of view will have a lasting influence.

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Introduction to modern prime number theory. By T. Estermann. Cambridge University Press, 1952. 10+75 pp. \$2.50.

The main purpose of this tract is "to enable those mathematicians who are not specialists in the theory of numbers to learn some of its non-elementary results and methods without too great an effort." Actually, the book is devoted to the limited object of proving the Vinogradov-Goldbach theorem that every sufficiently large odd number is the sum of three primes; in the course of proving this result, the author supplies the necessary results on characters and primes in arithmetic progressions. Only a few elementary number-theoretic results are assumed, these being quoted from Hardy and Wright's book *An introduction to the theory of numbers*; Cauchy's residue theorem is also assumed.

The book is a very carefully thought out exposition which lays bare the whole nature of the proof and unremittingly avoids all things not needed in the final proof. This leads to a work which is somewhat austere although not so formal as Landau's *Vorlesungen über Zahlentheorie*; in common with Landau's book, Estermann's tract gives few references to the literature. Nevertheless, the author admirably succeeds in his aim. The proofs are clear and remarkably

³ In Chapter III, the sole assertion resembling Fubini's theorem is stated for continuous functions with compact support only.

concise. Some idea of the conciseness may be gained from the fact that the only comparable work—that of N. G. Čudakov, in Russian, entitled *Introduction to the theory of Dirichlet's L-functions*, 1947—is a little under three times as long; the length of Čudakov's book is partly due to the fact that, unlike Estermann, he makes excursions to round out the background. (Actually, Čudakov's book is less complete than Estermann's in one respect; namely, the former does not contain a proof of a fundamental estimate of Vinogradov which occupies half of the last chapter of Estermann's book.)

One of the essential steps in the proof is to obtain a suitable estimate for $\pi(m; k, l)$ which is defined as the number of primes p between 1 and m which are of the form $kn+l$. Chapter I is devoted to proving the following result for $\pi(m) = \pi(m; 1, 1)$, the number of primes between 1 and m :

$$\pi(m) = \text{ls } m + O(m e^{-c(\log m)^{1/2}})$$

where $c = 1/200$ and

$$\text{ls } m = \sum_{n=2}^m 1/\log n = \int_2^m du/\log u + O(1) = \text{li } m + O(1).$$

This result is de la Vallée Poussin's refinement of the prime number theorem (later sharpened by Čudakov and others). The proof follows the ideas of Landau and only uses the Riemann zeta function $\zeta(s)$ for $\sigma = \Re(s) > 0$. Such function-theoretic results as the Borel-Carathéodory theorem, which gives an upper bound for $|f^{(n)}(0)|$ in terms of an upper bound for $\Re f(z)$ on the unit circle, are proved. These results lay the groundwork for the whole method of proof which is used both in this and the next chapter. The author's decision to obtain the above numerical value for c occasionally forces the reader to use a slide rule and leads to proofs in which the underlying reason for their success is somewhat obscured. The author, perhaps, partly recognizes this when he wryly remarks, after obtaining the contradiction $333 > 1000/3$, that this reminds him "of the story of the Scotsman who looked suspiciously at his change, and when asked if it was not enough, said: 'Yes, but only just'."

Chapter II begins with the development of the theory of characters χ modulo k . This is done with the aid of the basis theorem for finite abelian groups which is proved in the special case of the multiplicative group of reduced residue classes modulo k . The theory of $L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)/n^s$ is now developed for $\chi \neq \chi_0$, the principal character modulo k . That χ_0 can be disregarded is due to the fact that if $\pi(m; \chi) = \sum_{p \leq m} \chi(p)$, then $0 \leq \pi(m) - \pi(m; \chi_0) \leq \phi(k)$ and

$$\begin{aligned}\pi(m; k, l) &= \sum_{\chi \bmod k} \bar{\chi}(l) \pi(m; \chi) / \phi(k) \\ &= \pi(m) / \phi(k) + \sum_{\chi \neq \chi_0} \bar{\chi}(l) \pi(m; \chi) / \phi(k) + O(1).\end{aligned}$$

Because of the result of Chapter I, it suffices to deal with $\pi(m; \chi)$ for $\chi \neq \chi_0$. The usual zero-free regions of $L(s, \chi)$, roughly of the form $\sigma > 1 - C/\log kt$, are obtained. Also, the author gives his proof of Siegel's result that $L(s, \chi) \neq 0$ for real $s > 1 - k^{-\epsilon}$ if $k > A(\epsilon)$. This work is handled adroitly. No mention is made of the possible exceptional real (Siegel-) zero, nor is the idea of primitive character needed or mentioned; for a more perfect understanding of the underlying situation, this may be a loss but it enables the author to drive towards his goal more quickly. The final result, Theorem 55, is that

$$\pi(m; k, l) = \text{ls } m / \phi(k) + O(m e^{-c(\log m)^{1/2}})$$

if $(k, l) = 1$, $k \leq \log^u m$, $m \geq 3$, and u is an arbitrary real number on which O depends; the principal term arises from $\pi(m)$ while the error term arises from the functions $\pi(m; \chi)$.

Actually, this form of the result is not that needed by the author at the bottom of page 63. What is needed is the slightly more general result that if $(k, l) = 1$, $k \leq \log^u M$, $M \geq 3$, and $m \leq M$, then

$$\pi(m; k, l) = \text{ls } m / \phi(k) + O(M e^{-c(\log M)^{1/2}}).$$

This result can be obtained, for example, by first developing the corresponding form of Theorem 52; all that is required for this purpose is that b be defined as $\exp(\{\log(M+1/2)\}^{1/2}/160)$ and that the two cases $M > A_3$ and $M \leq A_3$ be considered. Also, once this is done, the first half of the proof of Theorem 54 may be dispensed with.

The Vinogradov-Goldbach theorem is proved in the last chapter by the usual Farey dissection of the unit circle. The most difficult and least transparent part of this proof (although technically elementary) is the following weakened form of the Vinogradov estimate:

$$\left| \sum_{p \leq v} e^{2\pi i p h/q} \right| \leq n / \log^3 n$$

provided $(h, q) = 1$ and

$$n / \log^3 n < v \leq n, \quad \log^{15} n < q \leq n / \log^{15} n.$$

The proof occupies eight pages and here too the author has effected a considerable simplification.

At the end of the book is an index and a seven page section of

theorems and formulae for reference. The book has the usual excellent typography of the Cambridge tracts and is almost free of misprints.

LOWELL SCHOENFELD

Sur les espaces fibrés et les variétés feuilletées. By W. T. Wu and G. Reeb. (Actualités Scientifiques et Industrielles, no. 1183.) Paris, Hermann, 1952. 157 pp.

This monograph consists of the theses of the authors at the University of Strasbourg under the direction of C. Ehresmann, and contains detailed accounts of original contributions of the authors, whose main results have been announced in the Comptes Rendus de l'Académie des Sciences à Paris. The contents of the two papers are not directly related, although both can be said to be concerned with certain aspects of the theory of differentiable manifolds.

The paper of Wu Wen-Tsun has the complete title: *Sur les classes caractéristiques des structures fibrées sphériques*. Its starting-point is the so-called universal bundle theorem. In the cases in which the author is interested, the fiber bundle has as base space a finite polyhedron B and as structural group G one of the three groups: the orthogonal group O_m in m variables, the proper orthogonal group \hat{O}_m in m variables, and the unitary group O'_m in m complex variables. The universal bundle theorem asserts that, B and G being given, there exists a bundle with the base space B_0 and the same structural group G such that the given bundle is induced by a mapping $f: B \rightarrow B_0$ and that this mapping f is defined up to a homotopy. It was perhaps Pontrjagin who first observed that the dual homomorphism $f^*: H(B_0, R) \rightarrow H(B, R)$ of the cohomology ring of B_0 into the cohomology ring of B , relative to a coefficient ring R , is thus completely determined by the bundle. In our three cases we can take as B_0 respectively the following Grassmann manifolds: the manifold $R_{n,m}$ of all m -dimensional linear spaces through a point O in a real Euclidean space of dimension $n+m$, the manifold $\widehat{R}_{n,m}$ of such oriented linear spaces, and the manifold $C_{n,m}$ of all linear spaces of dimension m through a point O in a complex Euclidean space of complex dimension $n+m$, with n sufficiently large in all three cases. The author's notation for $C_{n,m}$ is slightly confusing. In many places, such as on pages 7, 8, 48, etc., it would be clearer to write $C_{n',m'}$ for $C_{n,m}$.

The fruitfulness of this approach is based on the fact that these Grassmann manifolds are relatively "rich" in homology properties. The homology groups of these manifolds were determined by Ehresmann, on adopting the notion of Schubert varieties in algebraic geometry. Since f^* is a multiplicative homomorphism, it suffices to