algorithmic method. The results of this chapter are used in Chapter V to give an algorithmic treatment of various questions connected with finite systems of differential equations. For the case of a field $\mathcal{J}$ consisting of analytic functions, a very useful approximation theorem is proved.

Chapter V deals with constructive methods and tests.

In Chapter VI, the case of a field of analytic functions is treated by analytic methods. For this case, another proof of the low power theorem is given.

Chapter VII deals with intersections of algebraic differential manifolds, especially with their dimensions. A result of Jacobi proves true in some special cases, but false in general.

Chapters VIII and IX deal with partial differential equations. In Chapter VIII, a very important existence theorem, due to Riquier, is proved. In Chapter IX this theorem is used to extend some of the main results of the preceding chapters to partial differential polynomials.

B. L. VAN DER WAERDEN

Transcendental numbers. By Carl Ludwig Siegel. (Annals of Mathematics Studies, no. 16.) Princeton University Press, 1949. 8+102 pp. $2.00.

As the author states in a short preface, this book is based on lectures given at Princeton in 1946. In Chapter I, *The exponential function*, proofs are given of the irrationality of e and π, and then a general method is introduced.

Let $\rho_1, \cdots, \rho_m$ be complex numbers, $n_1, \cdots, n_m$, non-negative integers, and let

$$N + 1 = \sum_{k=1}^{m} (n_k + 1).$$

It is shown that polynomials $P_1(x), \cdots, P_m(x)$ of degrees $n_1, \cdots, n_m$, respectively, may be determined uniquely (up to a constant factor) such that the function

$$R(x) = \sum_{k=1}^{m} P_k(x)e^{\rho_k x}$$

vanishes at $x=0$ of order N. Such a function is called an approximation form. An explicit formula for $R(x)$ as a multiple integral provides an upper bound for $|R(1)|$ and shows that $R(1)>0$ when $\rho_1, \cdots, \rho_m$ are real.

The stage is then set for the proof that e^a is transcendental for real algebraic $a \neq 0$. If e^a were algebraic, there would be an algebraic integer ξ, not zero, but whose norm is less than one. This is clearly a contradiction. A similar argument, based on m approximation forms, leads to the Lindemann-Weierstrass theorem which states that, if $b_1, \cdots, b_r$ are different algebraic numbers, then $e^{b_1}, \cdots, e^{b_r}$, are not related by a homogeneous linear equation with algebraic coefficients not all zero.

The exponential function $y = e^x$ satisfies the differential equation $y' = y$ and has the addition theorem $e^x e^t = e^{x+t}$. The remaining chapters are concerned with functions that are either solutions of linear differential equations or possess addition theorems.

Chapter II, *Solutions of linear differential equations*, deals with what are called E-functions. A function $f(x)$ is an E-function if

$$f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!},$$

where the coefficients c_n belong to the same algebraic number field of finite degree over the rational number field, and where the maximum of the absolute values of all conjugates of c_n and the least common (rational) denominator of $c_1, \cdots, c_n$, regarded as functions of n, are each $O(n^{n\epsilon})$ for any $\epsilon > 0$. (The exponential function is clearly an E-function.)

Let $E_1, \cdots, E_m$ be E-functions that are solutions of a system of m homogeneous linear differential equations of the first order,

$$y_k' = \sum_{j=1}^{m} Q_{kj}(x) y_j \qquad (k = 1, \cdots, m),$$

whose coefficients Q_{kj} are rational functions of x with algebraic numerical coefficients. If the products $E_1^{\nu_1} \cdots E_m^{\nu_m}$, $\nu_1 + \cdots + \nu_m \leqq \nu$, form what is called a normal system for all positive integral ν, and if α is any algebraic number different from zero and the poles of Q_{kj}, it is proved that $E_1(\alpha), \cdots, E_m(\alpha)$ are not related by an algebraic equation with algebraic coefficients. This result includes the Lindemann-Weierstrass theorem.

A particular E-function is

$$K_\lambda(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!(\lambda+1)\cdots(\lambda+n)2^{2n}}, \qquad \lambda \neq -1, -2, \cdots,$$

which is related to the Bessel function $J_\lambda(x)$ by the equation

$$J_\lambda(x) = \frac{1}{\Gamma(\lambda + 1)} \frac{x^\lambda}{2^\lambda} K_\lambda(x).$$

It is shown that the normality condition is satisfied for all rational $\lambda \neq \pm 1/2, \pm 3/2, \cdots$. With these restrictions, it follows that $K_\lambda(\alpha)$ is transcendental for any algebraic $\alpha \neq 0$. Hence all the zeros of $K_\lambda(x)$ and all the zeros not equal to 0 of $J_\lambda(x)$ are transcendental.

The function

$$w = f_\lambda(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!(\lambda + 1) \cdots (\lambda + n)}$$

satisfies the differential equation $xw'' + (\lambda + 1)\omega' = w$, which leads to the continued fraction

$$\frac{w}{w'} = \lambda + 1 + \frac{x}{\lambda + 2 + x/(\lambda + 3 + \cdots)} \cdot$$

Since $f_\lambda(-x^2/4) = K_\lambda(x)$, the previous results show that the continued fraction is transcendental for all algebraic $x \neq 0$ and all rational λ. In particular,

$$1 + \cfrac{1}{2 + \cfrac{1}{3 + \cdots}}$$

is transcendental.

Chapter III, *The transcendency of a^b for irrational algebraic b and algebraic $a \neq 0$*, is a short one giving Schneider's proof and a simplified form of Gelfond's. Each proof is similar in principle to the proof of the transcendence of e^α, but is naturally much deeper.

Chapter IV, *Elliptic functions*, deals with some results due to Schneider. Let $\wp(z)$ and $\zeta(z)$ denote the usual Weierstrass functions, where $\zeta'(z) = -\wp(z)$. It is shown that, if λ, μ, g_2, g_3, $\wp(z_0)$ are algebraic and λ, μ are not both zero, then $\lambda z_0 + \mu\zeta(z_0)$ is transcendental. As a special case it follows, for example, that the arc length of the lemniscate $(x^2+y^2)^2 = 2a^2(x^2 - y^2)$ is transcendental for algebraic a between points with algebraic coordinates.

It can be seen from the foregoing paragraphs that the book deals authoritatively with the problems of transcendental numbers. Unfortunately, it is not an easy book to follow. The subject matter is admittedly difficult, but, in the opinion of the reviewer, much could be done to present the material in a way that is clear and well organized.

The Editors of the series and Princeton University Press must share the blame for the poor form and style of the book. Many of the lines in the text are badly set up, some symbols are all too obviously handwritten, and the formulas are generally difficult to read.

The organization of the book leaves much to be desired, and, in the opinion of the reviewer, the fault lies in the method of presenting a proof. The author seldom takes the trouble to let the reader know what he is doing and why. This is clearly illustrated in the first chapter where a proof is given of the transcendency of e^a. The reader may find the proofs involved, but not until the end of the chapter, if he gets that far, does he discover that the author has a perfectly good reason for presenting the proofs in the way he does.

Pólya, writing in the American Mathematical Monthly, December 1949, page 684, describes the situation exactly in the following words. "A mathematical lecture should be, first of all, correct and unambiguous. Still, we know from painful experience that a perfectly unambiguous and correct exposition can be far from satisfactory and may appear uninspiring, tiresome or disappointing, even if the subject-matter presented is interesting in itself."

R. D. JAMES

Fourier transforms. By S. Bochner and K. Chandrasekharan. (Annals of Mathematics Studies, no. 19.) Princeton University Press, 1949. 10+219 pp. $3.50.

This is a very readable introduction to the craft of the authors, and as such fills a very real need. The subject matter serves to a large extent as a springboard for the presentation of interesting techniques, viewpoints, and concepts, and is treated with great deftness and remarkable continuity. There is a good deal of explanatory and motivating material, and altogether the book is a very appropriate one for study by an apprentice to the guild of semi-classical analysts.

The book is apparently not intended for reference use, nor is its subject matter and development the sort that are best adapted to the needs of a mathematician with a merely general interest in the subject, or to the needs of a theoretical physicist. The topics treated are interesting but the basis of their selection seems to have been esthetic and subjective, rather than a function of their relative significance within the general framework of mathematics. Among the important topics not treated are Fourier transforms in the complex domain, Fourier-Stieltjes transforms, and generalized harmonic analysis; from