A NONHOMOGENEOUS MINIMAL SET

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1. Introduction. In this note we consider the following question: *does there exist a compact minimal set which is of dimension* 0 *at some of its points and of positive dimension at others?* We answer the question in the affirmative by constructing a compact plane set X and a homeomorphism T of X onto X such that X is minimal with respect to T (that is, contains no proper closed subset Y with $T(Y) \subset Y$) and such that X possesses the desired property. As a result, there exist nonhomogeneous minimal sets.

An outline of the procedure is as follows. A compact, totally disconnected subset A of the x-axis in the plane and a homeomorphism f of A onto A are defined so that A is minimal with respect to f. Two real functions b_0 and b_1 are then defined on A with $0 \leqq b_0(x) \leqq b_1(x) \leqq 1$. We then let X be the set of all points $(x, tb_1(x)+(1-t)b_0(x))$ for $x \in A$ and $0 \leqq t \leqq 1$, thus in effect erecting a vertical interval or a point over each $x \in A$. Then T is defined so as to send the point determined by x and t into the point determined by $f(x)$ and t.

2. The example.

DEFINITIONS. Let A_i denote the set of integers $1, \cdots, 3^i$ and let π_{i+1} be the map from A_{i+1} to A_i defined by $\pi_{i+1}(p) = p \bmod 3^i$ for $p \in A_{i+1}$. Let A designate the limit space of the sequence (A_i, π_{i+1}) [1].[1] Then A is a compact totally disconnected metric space. Let f_i be the map from A_i onto A_i defined by $f_i(p) = (p+1) \bmod 3^i$; then $\pi_{i+1}f_{i+1} = f_i\pi_{i+1}$. It follows that the map defined by $f(x) = f_i((x_i))$ for $x = (x_i) \in A$ is a homeomorphism of A onto A. Moreover A is minimal with respect to f for if $x = (x_i) \in A$ and $y = (y_i) \in A$, then $f^{y_n - x_n}(x)$ has its first n coordinates equal to those of y.

Let $x = (x_i) \in A$; the points of A_{i+1} mapping onto x_i under π_{i+1} are $x_i + \alpha \cdot 3^i$, $\alpha = 0, 1, 2$. Define α_i by $x_{i+1} = x_i + \alpha_i \cdot 3^i$. We call the subsequence $\beta_1, \beta_2, \cdots$ of $\alpha_1, \alpha_2, \cdots$ consisting of all $\alpha_i \neq 1$ the *associated sequence* for x, and define several functions of x:

Let $a(x)$ be the number of elements in the associated sequence for x ($a(x)$ is either a non-negative integer or ∞).

Let $b_0(x) = 0$ *if* $a(x) = 0$, $b_0(x) = (1/2) \sum_{j=1}^{a(x)} \beta_j/2^j$ *if* $a(x) > 0$.

Let $b_1(x) = b_0(x) + \sum_{j > a(x)} 1/2^j = b_0(x) + 1/2^{a(x)}$ if $a(x) < \infty$, *and let*

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[1] Numbers enclosed in brackets refer to the bibliography at the end of the paper.

$b_1(x)=b_0(x)$ *if* $a(x)=\infty$.

Let $b_t(x)=tb_1(x)+(1-t)b_0(x)$ *for* $0\leqq t\leqq 1$.

Since A is a totally disconnected compact metric space, it may be embedded in the x-axis of the plane. Let X be the set of all points $(x, b_t(x))$, $x\in A$ and $0\leqq t\leqq 1$. Define a transformation T of X into X by $T((x, b_t(x)))=(f(x), b_t(f(x))$.

THEOREM. *The space X is compact and of dimension* 0 *at some points and of dimension* 1 *at the others. Furthermore T is a homeomorphism of X onto X and X is minimal with respect to T.*

PROOF. The proof will be divided into four parts.

(a) *X is compact.* We state the following, omitting proof: the function $b_0(x)$ is lower semi-continuous on A and $b_1(x)$ is upper semi-continuous on A. Let $(x_j, b_{t_j}(x_j))$ be an infinite sequence of points in X. Some subsequence, which we suppose the same as the original, converges to a point $z=(x, y)$ of the plane. Since A is compact we have $x\in A$. For $y\in A$ let $V(y)=\bigcup_{0\leqq t\leqq 1}(y, b_t(y))$. By the semi-continuity statements we have lim sup $V(x_i)\subset V(x)$. But $(x_i, b_t(x_i))\in V(x_i)$ and hence $z\in V(x)\subset X$.

(b) *X is of dimension* 0 *at some of its points and dimension* 1 *at the others.* Since A is totally disconnected, the components of X are the sets $V(x)$, $x\in A$. If $a(x)=\infty$, then $V(x)$ is a single point and the dimension of X at x is 0. If $a(x)$ is finite then $V(x)$ is a line segment and the dimension of X at x is greater than or equal to 1. This dimension is exactly 1 since X is the subset of a product of a space of dimension 0 and a space of dimension 1.

(c) *T is a homeomorphism of X onto X.* Let $x=(x_i)\in A$. Suppose for some m $x_{m+1}\neq 3^{m+1}$; let m be the first such integer. We say that x is of *type* 1 if $m=0$; if $m>0$ we say that x is of *type* 2 or 3 according as α_m is 0 or 1. Then:

(i) If x is of type 1, $a(f(x))=a(x)$, $b_t(f(x))=b_t(x)$.

(ii) If x is of type 2, $a(f(x))=a(x)-1$; $b_t(f(x))=2b_t(x)-2+1/2^{m-2}$.

(iii) If x is of type 3, $a(f(x))=a(x)+1$, $b_t(f(x))=(1/2)b_t(x)-1/2+1/2^{m-1}$.

Of the above statements we prove (ii) as typical. We then have $x=(x_i)$ where $x_i=3^i$ for $i\leqq m$, $x_{m+1}=3^m$, $x_i\neq 3^i$ for $i\geqq m+1$. The associated sequence is then $2, \cdots, 2, 0, \beta_{m+1}, \cdots$ led by $(m-1)$ 2's. Then

$$(*)\quad b_0(x)=\sum_{j=1}^{m-1}1/2^j+\sum_{j\geqq m+1}\beta_j/2^{j+1}=1-1/2^{m-1}+\sum_{j\geqq m+1}\beta_j/2^{j+1}.$$

Also $f(x)=(y_i)$ where $y_j=1$ for $i\leqq m$, $y_{m+1}=1+3^m$, $y_i=1+x_i$ for

$i \geqq m+1$. The associated sequence of $f(x)$ is then $0, \cdots, 0, \beta_{m+1}, \cdots$ led by $(m-1)$ 0's and hence $b_0(f(x)) = \sum_{j \geqq m+1} \beta_j/2^j$. Solving this with (*) we obtain $b_0(f(x)) = 2b_0(x) - 2 + 1/2^{m-2}$. Also, by inspecting the associated sequences, it follows that $a(f(x)) = a(x) - 1$. These formulas imply the result for $b_t(f(x))$.

Inspection of these formulas together with consideration of the one exceptional point shows that $a(f(x))$ is infinite if and only if $a(x)$ is infinite. Then $V(f(x))$ is a point if and only if $V(x)$ is a point. Since f is 1–1 and onto, T is then 1–1 and onto X.

The continuity of T at $(x, b_t(x))$ follows from the formulas in case x is one of the types; for the exceptional point continuity may be checked directly.

(d) *X is minimal with respect to T.* Consider the point $x = (1, 1, \cdots) \in A$ and the point $z = (x, 0) \in X$. We show that z is an almost periodic point[2] under T and that $\bigcup_{n=-\infty}^{+\infty} T^n(z)$ is dense in X. It will then follow that X is minimal under T. To show z almost periodic, we notice that $V(x) = z$; that is, X is of dimension 0 at z. Then there exist arbitrarily small neighborhoods U of x in A and V of z in X such that $(y, b_t(y)) \in V$ if and only if $y \in U$. But then almost periodicity of x under f implies the almost periodicity of z under T.

Let $w = (y, b_t(y)) \in X$ where $y = (y_i) \in A$. We must now define a sequence (k_n) of integers with $T^{k_n}(z) \to w$. Since $b_0(y) \leqq b_t(y) \leqq b_1(y)$, it follows from the definitions that the associated sequence $(\beta_j | j = 1, \cdots, a(y))$ may be extended by addition of elements to obtain a sequence $(\beta_j | j = 1, \cdots, \infty)$, each β_j either 0 or 2, such that $b_t(y) = \sum_{j=1}^{\infty} \beta_j/2^{j+1}$. Define α_i by $y_{i+1} = y_i + \alpha_i \cdot 3^i$. Let $n > 0$; let m be the number of elements in $\alpha_1, \cdots, \alpha_{n-1}$ which are not 1. Let $y'_j = y_j$ for $j \leqq n$, $y'_{n+j+1} = y'_{n+j} + \beta_{m+j+1} \cdot 3^{n+j}$ for $0 \leqq j \leqq n-1$, and $y'_j = y'_{2n}$ for $j \geqq 2n$. By construction, $y^{(n)} = (y'_j)$ is a point of A. Since from some point on the coordinates of $y^{(n)}$ are identical, then $k_n = y'_{2n} - 1$ has the property that $f^{k_n}(x) = y^{(n)}$. The associated sequence of $y^{(n)}$ is $\beta_1, \beta_2, \cdots, \beta_m, \cdots, \beta_{m+n}, 0, \cdots$ and hence $b_0(y^{(n)}) = \sum_{j=1}^{m+n} \beta_j/2^{j+1}$. Then $y^{(n)} \to y$ and $b_0(y^{(n)}) \to b_t(y)$ as n increases. Then $T^{k_n}(z) = (y^{(n)}, b_0(y^{(n)}))$ which converges to w as n increases. This completes the proof.

REMARK. Let P be a topological space and let g be a homeomorphism from P onto itself. Then a point $p \in P$ is said to be regularly almost periodic if and only if for each neighborhood V of p there exists a positive integer k such that the closure of $\bigcup_{n=-\infty}^{+\infty} g^{nk}(p)$

[2] The point z is said to be almost periodic under T if to $e > 0$ there corresponds a positive integer N such that in every set of N consecutive integers appears an integer n with $\rho(z, T^n(z) < e$.

is contained in V. An example has been given by Garcia and Hedlund [2] of a minimal set some of whose points are regularly almost periodic while others are not. The example of the preceding theorem also has this property. It is easily verified that each point of A is regularly almost periodic with respect to f; it follows that points of X at which the dimension is 0 are also regularly almost periodic. All points of X, however, are not regularly almost periodic, for otherwise a theorem of Garcia and Hedlund [2] would give us that T has equicontinuous powers, which is false.

BIBLIOGRAPHY

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