TRANSCENDENCE OF FACTORIAL SERIES WITH PERIODIC COEFFICENTS

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It is well known that every real number α can be represented by a factorial series

$$\alpha = \frac{a_1}{1!} + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots + \frac{a_n}{n!} + \cdots, \tag{1}$$

where the a_n ($n=1, 2, \cdots$) are integers and, moreover, $0 \leqq a_n < n$ (for $n=2, 3, \cdots$). This representation is unique for the irrational numbers α, while every rational α can be represented either with almost all[1] $a_n = 0$ or with almost all $a_n = n-1$.

The representation (1) was discussed and the aforesaid properties were proved by M. Stéphanos [1].[2] But an even more general type of series had already been studied by G. Cantor [2] (not known to M. Stéphanos). These series have later been called "Cantor series" (cf. [3]).

In this note we consider the case in which the factorial series (1) has periodic coefficients a_n and we prove the following theorem:

THEOREM 1. *Every number α represented by a factorial series* (1) *with periodic coefficients is transcendental* (*except for the trivial case where almost all a_n are zero*).

The above condition $0 \leqq a_n < n$ (for $n=2, 3, \cdots$) is not used at all in the following proof. Moreover, the coefficients a_n need not necessarily be integers; the a_n can be *any algebraic numbers*. Then Theorem 1 and its proof still hold.

We generalize Theorem 1 further:

THEOREM 2. *If the power series*[3]

$$\phi(z) = \sum_{n=1}^{\infty} \frac{a_n}{n!} z^n \tag{2}$$

has algebraic coefficients a_n (not almost all of them being zero) which form a periodic sequence, then $\phi(z)$ is a transcendental number for every

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[1] The expression "almost all" is used in the sense of "all but a finite number."

[2] Numbers in brackets refer to the bibliography at the end.

[3] Under the conditions of Theorem 2, $\phi(z)$ is an entire function.

algebraic $z(\neq 0)$.

For $z=1$ Theorem 2 furnishes Theorem 1. Hence we prove Theorem 2.

The general case in which the period starts with a_s $(s\geqq 1)$ can immediately be reduced to the case in which the period begins with a_1. One has only to subtract and add $s-1$ terms, that is, algebraic numbers. Therefore it suffices to assume that the period starts at a_1.

Let m be the "length" of the period, that is, the number of coefficients a_n belonging to the period, and let ω be a primitive mth root of unity. Then

$$e^{\omega k\cdot z} = 1 + \frac{\omega^k\cdot z}{1!} + \frac{\omega^{k2}\cdot z^2}{2!} + \cdots + \frac{\omega^{k(m-1)}\cdot z^{(m-1)}}{(m-1)!} + \frac{\omega^{km}\cdot z^m}{m!} + \cdots.$$

Thus, since the sum of the rth powers of all the mth roots of unity is zero if r is not divisible by m, we obtain:

$$\sum_{k=1}^{m} e^{\omega k\cdot z} = m + 0 + \cdots + 0 + m\frac{z^m}{m!} + 0 + \cdots + 0 + m\frac{z^{2m}}{(2m)!} + \cdots$$

and hence

$$\frac{a_m}{m}\sum_{k=1}^{m} e^{\omega k\cdot z} = a_m + 0 + \cdots + 0 + a_m\frac{z^m}{m!} + 0 + \cdots + 0 + a_m\frac{z^{2m}}{(2m)!} + 0 + \cdots.$$

Similarly for $r=1, 2, \cdots, m-1$ we have

$$\frac{a_{m-r}}{m}\sum_{k=1}^{m} \omega^{kr}e^{\omega k\cdot z} = 0 + \cdots + 0 + a_{m-r}\frac{z^{m-r}}{(m-r)!} + 0 + \cdots + 0 + a_{m-r}\frac{z^{2m-r}}{(2m-r)!} + 0 + \cdots.$$

By summing over $r=0, 1, 2, \cdots, m-1$, we obtain

$$\sum_{k=1}^{m} e^{\omega k\cdot z}\left(\frac{1}{m}\sum_{r=0}^{m-1} a_{m-r}\omega^{kr}\right) = \phi(z) + a_m$$

or, if we set $(1/m)\sum_{r=0}^{m-1} a_{m-r}\omega^{kr} = A_k$ $(k=1, 2, \cdots, m)$, we have

$$\sum_{k=1}^{m} A_k e^{\omega^k \cdot z} - [\phi(z) + a_m]\cdot e^0 = 0. \tag{3}$$

It is impossible that all coefficients A_k $(k=1, 2, \cdots, m)$ vanish; for then the equation of $(m-1)$st degree

$$\sum_{r=0}^{m-1} a_{m-r} z^r = 0$$

would have m different roots ω^k $(k=1, 2, \cdots, m)$, while the case that all $a_n=0$ $(n=1, 2, \cdots, m)$ has been excluded. But now, since for an algebraic $z\neq 0$ the numbers $\omega^k\cdot z$ $(k=1, 2, \cdots, m)$ are algebraic, different, and nonzero and since the A_k $(k=1, 2, \cdots, m)$ are also algebraic numbers and at least one of them is not zero, it follows from (3) by Lindemann's general theorem ([**4**, **5**], cf. also [**6**]) that $\phi(z) + a_m$ and hence also $\phi(z)$ is a transcendental number for every algebraic $z \neq 0$.

Bibliography

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