is a projective collineation of order $N_{n,0}^m$ and leaves invariant $(z_0^{p^{im}}, z_1^{p^{im}}, \cdots, z_n^{p^{im}})$, $i=0, 1, \cdots, n$, and the given S_n^m. Therefore (3) is a power of T. Since a fixed S_n^r of (3) has been obtained, on denoting it by R_n^r we have that

$$T^i(R_n^r), \qquad i = 1, 2, \cdots, N_{n,0}^m/N_{n,0}^r,$$

are the fixed S_n^r of (3), where $T^i(R_n^r)$ represents the image of R_n^r effected by T^i. These $N_{n,0}^m/N_{n,0}^r$ fixed S_n^r evidently satisfy the condition of the theorem. Thus we have completed the proof.

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SOME CONSEQUENCES OF A WELL KNOWN THEOREM ON CONICS

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Graustein [4, p. 296][1] proves the following theorem:

THEOREM I. *If three point conics have a common chord, and the three conics are taken in pairs and the common chord of each pair which is opposite to the given common chord is drawn, the three resulting lines are concurrent.*

He remarks that several well known theorems, including those of Pascal and the existence of the radical center of 3 non-coaxal circles, are obtainable as special cases of the above. The following result also follows directly from Theorem I:

COROLLARY 1. *The joins of the intersections of the opposite sides of a complete quadrangle with a conic passing through two vertices of the quadrangle are concurrent.*

This corollary furnishes a simple proof of Ex. 155, p. 307 of Baker [1]: Let A, B, C, O be 4 points of a conic; let a line meet BC, CA, AB respectively in L, M, N; and OL, OM, ON meet the conic again in P, Q, R respectively. Then AP, BQ, CR meet in a point, lying on the line LMN.

It does not seem to have been noted that the following theorem may be obtained directly from Theorem I.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

THEOREM 1. *The intersections of each of the conics of a pencil with a fixed conic* $\mathcal{C}$ *passing through* 2 *of the base points of the pencil are pairs of points of an involution on* $\mathcal{C}$.

PROOF. Suppose that the base points of the pencil of conics are A, B, C, D; that $\mathcal{C}$ passes through A and B; and that $\mathcal{C}$ intersects any 3 conics Σ_1, Σ_2, Σ_3 of the pencil in P_1, Q_1; P_2, Q_2; and P_3, Q_3. Then, by Theorem I, the lines P_1Q_1, P_2Q_2, CD are concurrent; so also are the lines P_1Q_1, P_3Q_3, CD. Hence the lines P_iQ_i all pass through the same point, for all points of intersection P_i,Q_i of $\mathcal{C}$ with an arbitrary conic, Σ_i, of the pencil. But this last is just the condition that P_i, Q_i are pairs of an involution on $\mathcal{C}$.

This result might also have been obtained more laboriously (for the proof of Theorem I itself is easy) by the method which Baker [2, pp. 134–138] uses for general involutions.

The involution theorem of Desargues follows from Theorem 1 as a special case by letting $\mathcal{C}$ consist of the line AB and a line not through any of the points A, B, C, D.

Theorem I leads to a poristic result in the following manner: Let Σ, Σ' be two conics through A, B, C, D. Choose A_1 on Σ, and draw AA_1 to meet Σ' in α_1. Let α_1B meet Σ in A_2; AA_2 meet Σ' in α_2, and so on. Starting with A_1 on Σ, we thus set up a sequence of points, A_i, on Σ: AA_i meets Σ' in α_i, and α_iB meets Σ in A_{i+1}. Then we have the following theorem.

THEOREM 2. *If* A_n *coincides with* A_1 *for some position of* A_1, *then* A_n *will coincide with* A_1 *for all positions of* A_1.

PROOF. Choose B_1 on Σ and let the intersection of BB_1 with Σ' be called β_1. The lines AA_1 and BB_1 constitute a conic through A and B; hence, by Theorem I, A_1B_1, $\alpha_1\beta_1$, and CD are collinear, at O, say. If the intersection of β_1A with Σ is called B_2, then, by the same argument, A_2B_2 passes through O. (We have, then, two sets of lines, A_iB_i and $\alpha_i\beta_i$, all concurrent at O). Now, if, for some A_1, A_n coincides with A_1, B_n will coincide with B_1, for all B_1. Hence, in turn, A_n will coincide with A_1, for all A_1.

The condition for periodicity when the two conics are circles may be easily obtained:

THEOREM 3. *Let* S, S' *be two circles, centered at* O, O', *intersecting at* A *and* B. *The sequence* $\{A_i\}$ *on* S *is obtained as follows*: A_iA *intersects* S' *at* α_i; α_iB *intersects* S *at* A_{i+1}. *Then a necessary and sufficient condition that* A_n *shall coincide with* A_1 *for all starting points*, A_1, *on* S *is that angle* $OAO' = m\pi/n$ *where* m *and* n *have no common factor.*

PROOF. In triangle $A_iB\alpha_i$, each of the angles A_i and α_i is constant, because they are inscribed angles subtending the fixed arcs AB and BA of circles S and S'. Hence, the third angle, B, is also constant. Now when A_i has the particular position where A_iA is perpendicular to AB, then BA_i and $B\alpha_i$ are diameters; so that for this position angle $A_iB\alpha_i$ is angle OBO'. Hence angle $A_iB\alpha_i$ equals angle OBO' for all positions of A_i. Since angle $A_iB\alpha_i$ is the same as angle ABA_{i+1}, and since angle A_iBA_{i+1} equals 1/2 arc A_iA_{i+1}, it follows that arc A_iA_{i+1} equals twice angle OBO', for all positions of A_i.

The condition for A_n to coincide with A_1 is obviously that $\sum_1^n$ arc $A_iA_{i+1}=2m\pi$, or that angle OAO' (=angle OBO') $=m\pi/n$.

This result is related to Steiner's porism on a ring of circles tangent to two fixed circles and to the neighboring circles of the ring. (See, for example, Coolidge [3, p. 30] or Johnson [5, p. 115].)

COROLLARY 2. *Given two circles S_1, S_2 with their line of centers intersecting S_1 in P_1Q_1 and S_2 in P_2Q_2. Let the circles constructed on P_1Q_2 and P_2Q_1 as diameters be Σ_1, Σ_2. Then, if either of the pairs (S_1, S_2), (Σ_1, Σ_2) is a Steiner pair, the other is a pair of the type of Theorem 3, and conversely.*

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