SOME PROBABILITY LIMIT THEOREMS

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1. **Introduction.** We prove the following theorems:

Let $X_1, X_2, \cdots$ be independent, identically distributed random variables, each having mean 0 and standard deviation 1 and let $S_k = X_1 + X_2 + \cdots + X_k$.

I. If $N = [\alpha n]$ where $0 < \alpha < 1$, then:

$$\lim_{n\to\infty} \text{Prob}\left\{\min(S_{N+1}, S_{N+2}, \cdots, S_n) > \beta n^{1/2}\right\} = V_1(\beta)$$

where

$$V_1(\beta) = \frac{1}{\pi}\int_{u=\beta\alpha^{-1/2}}^{\infty} e^{-u^2/2}du \int_{t=0}^{(u\alpha^{1/2}-\beta)(1-\alpha)^{-1/2}} e^{-t^2/2}dt.$$

II. If $p(t)$ is continuous in $0 \leqq t \leqq 1$ and has at most a denumerable set of zeros in the interval, then:

$$\lim_{n\to\infty} \text{Prob}\left\{\frac{1}{n^2}\sum_{j=1}^{n} p(j/n)S_j^2 < \beta\right\} = V_2(\beta)$$

where the characteristic function of $V_2(\beta)$ is given by

$$\int_{-\infty}^{\infty} e^{i\beta\xi}dV_2(\beta) = (D(2i\xi))^{-1/2}$$

and $D(\lambda)$ is the Fredholm determinant associated with the integral equation

$$\lambda\int_0^1 \min(s, t)p(t)f(t)dt = f(s).$$

In the case where $p(t) > 0$ in $0 \leqq t \leqq 1$, we have

$$\int_{-\infty}^{\infty} e^{i\beta\xi}dV_2(\beta) = \prod_{j=1}^{\infty}(1 - 2i\xi\lambda_j)^{-1/2}$$

where $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of

$$g''(s) + p(s)g(s) = 0$$

subject to the conditions

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$$g(0) = g'(1) = 0.$$

We also prove the following:

III. Let $X_{n1}, X_{n2}, \cdots, X_{nn}$ be independent, identically distributed random variables, each having standard deviation 1 and mean μ_n where $\lim_{n\to\infty} n^{1/2}\mu_n = u$. Let $S_{nj} = X_{n1} + X_{n2} + \cdots + X_{nj}$. Then:

$$\lim_{n\to\infty} \text{Prob} \left\{ \frac{1}{n^2} \sum_{j=1}^{n} S_{nj}^2 < \beta \right\} = V_3(\beta)$$

where the characteristic function $V_3(\beta)$ is given by

$$\int_0^\infty e^{i\beta\xi} dV_3(\beta) = (\text{sech } (-2i\xi)^{1/2})^{1/2} \exp \left\{ \frac{\mu^2}{2} \left(\frac{\tanh (-2i\xi)^{1/2}}{(-2i\xi)^{1/2}} - 1 \right) \right\}.$$

The pattern of proof is essentially the same for the three theorems. We show first that the limiting distribution, if it exists, is independent of the distribution of the X's. A particular distribution is then chosen so that the limiting distribution can be conveniently calculated. This method was first used in a joint paper by Erdös and Kac [1].[1] Since our independence proofs differ only slightly from those in the latter paper, the details will not be given here.[2] Moreover, it will easily be seen that the requirement of identically distributed variables may be relaxed since it is only essential that the central limit theorem apply to the variables in question.

From I, the following corollary is obtained:

$$\lim_{n\to\infty} \text{Prob} \left\{ \min (S_{N+1}, \cdots, S_n) > 0 \right\} = \frac{1}{\pi} \sin^{-1} \alpha^{1/2}.$$

If, in addition, we let $M = [\gamma n]$, $0 < \gamma < \alpha < 1$, it can be shown by a method similar to the one used in proving I that

$$\lim_{w\to 0} \lim_{n\to 0} \text{Prob} \left\{ \min (S_{M+1}, S_{M+2}, \cdots, S_N) > 0 \Big| \, |S_n| < wn^{1/2} \right\}$$
$$= \frac{1}{\pi} \sin^{-1} \left(\frac{\gamma(1-\alpha)}{\alpha(1-\gamma)} \right)^{1/2}.$$

These special results had previously been derived by Lévy [3]. As stated, his theorems apply only to normally distributed X_i although he seems to have been aware that his theorems were true for the general class of distributions considered here.

Theorems II and III are generalizations of the third theorem in [1]. The calculation made in II for the special case where $p(t) > 0$ is similar to that made by Kac and Siegert [4, pp. 392–393]. There is a

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] The details may also be seen in another paper by the same authors [2].

footnote in the latter paper that in the general case we must resort to the Fredholm determinant but the details are not carried out there. The results of the special case of positive $p(t)$ in II and also III had been obtained by Cameron and Martin [5, 6] in their work on Wiener space. Their results are equivalent to ours for the case of normally distributed X's.

The theorems can easily be interpreted in terms of a one-dimensional random walk. Theorems I and II would apply to a free particle and III would apply to a particle in a constant force field.

2. Proof of I. Let

$$P_n(\beta) = \text{Prob}\left\{\min(S_{N+1}, \cdots, S_n) > \beta n^{1/2}\right\}.$$

Let k be a positive integer and let

$$n_j = N + [(n - N)j/k)] \qquad (j = 1, 2, \cdots, k)$$

and

$$P_{n,k}(\beta) = \text{Prob}\left\{\min(S_{n_1}, S_{n_2}, \cdots, S_{n_k}) > \beta n^{1/2}\right\}.$$

Then, for any $\epsilon > 0$, we have

$$P_{n,k}(\beta+\epsilon) - 1/k\epsilon^2 < P_n(\beta) < P_{n,k}(\beta).$$

Letting $n \to \infty$ and using the multidimensional central limit theorem, we have

$$\begin{aligned} &\text{Prob}\left\{\min(\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_k) > (\beta + \epsilon)k^{1/2}\right\} - 1/k\epsilon^2 \\ &\qquad \leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \\ &\qquad \leqq \text{Prob}\left\{\min(\overline{R}_1, \cdots, \overline{R}_k) > \beta k^{1/2}\right\} \end{aligned} \tag{1}$$

where

$$\begin{aligned} \overline{R}_1 &= (1 + (k - 1)\alpha)^{1/2} G_1, \\ \overline{R}_j &= \overline{R}_1 + (1 - \alpha)^{1/2}(G_2 + G_3 + \cdots + G_j) \qquad (j \geqq 2) \end{aligned}$$

and $G_1, G_2, \cdots$ are independent, normally distributed random variables each with mean 0 and standard deviation 1.

We now consider the particular case in which $X_1, X_2, \cdots$ are independent random variables such that

$$\text{Prob}\left\{X_j = 1\right\} = \left\{\text{Prob } X_j = -1\right\} = 1/2$$

for each j. We next obtain estimates on $P(n, l)$ for large n where

$$P(n, l) = \text{Prob}\left\{\min(S_1, S_2, \cdots, S_n) > -l\right\}$$

and l is an integer. For $l > 0$, we make use of a reflection argument to show that

$$\begin{aligned}\text{Prob}\,\{S_1 > -l, S_2 > -l, \cdots, S_{n-1} > -l, S_n = -l+p\}\\ = \text{Prob}\,\{S_n = -l+p\} - \text{Prob}\,\{S_n = -l-p\}\end{aligned}$$

and it will easily follow that

$$P(n, l) = 2\sum_{j=0}^{l} \text{Prob}\,\{S_n = j\} - \text{Prob}\,\{S_n = 0\} - \text{Prob}\,\{S_n = l\}.$$

We have

$$\begin{aligned}&\text{Prob}\,\{S_n = -l+p\}\\ &\quad= \text{Prob}\,\{S_1 > -l, \cdots, S_{n-1} > -l, S_n = -l+p\}\\ &\qquad+ \text{Prob}\,\{S_n = -l+p \text{ with } S_i = -l \text{ for at least one } i < n\}.\end{aligned}$$

We consider the third probability in this equality and we suppose that S_i is the first such sum for which $S_i = -l$. Since

$$\begin{aligned}\text{Prob}\,\{X_{i+1} + X_{i+2} + \cdots + X_n = p\}\\ = \text{Prob}\,\{X_{i+1} + \cdots + X_n = -p\},\end{aligned}$$

we have

$$\begin{aligned}\text{Prob}\,\{S_n = -l+p \text{ with } S_i = -l \text{ for at least one } i < n\}\\ = \text{Prob}\,\{S_n = -l-p\}\end{aligned}$$

and the desired result follows immediately.

We now make use of the following well known result [7, p. 135]:

$$\text{Prob}\,\{S_n = j\} = \begin{cases}(2/\pi n)^{1/2}e^{-j^2/2n} + \Delta & \text{if} \quad n \equiv j \pmod 2,\\ 0 & \text{otherwise}\end{cases}$$

where $|\Delta| < 3n^{-3/2}$ provided that $n \geqq 100$. Thus, for sufficiently large n, we have[3]

$$P(n, l) = 2\sum_{j=0}^{l}{}^{*}\left(\frac{2}{\pi n}\right)^{1/2} e^{-j^2/2n} + \epsilon_1 \qquad (l > 0)$$

where $|\epsilon_1| < 8ln^{-3/2} + 2n^{-1/2}$. Since

$$P(n, 0) = P(n-1, 1)/2$$

we have

$$P(n, 0) < 7n^{-1/2}.$$

Moreover, since

$$\text{Prob}\,\{S_1 > -l\} = 0 \qquad (l \leqq -1),$$

[3] An asterisk on a summation sign will indicate that only those values of the index of summation are taken for which the index and the upper limit are congruent modulo 2.

we have

$$P(n, l) = 0 \qquad (l < 0).$$

Now

$$P_n(\beta) = \sum_{l=-N}^{N} \text{Prob}\, \{S_N = l, \min (S_{N+1} - S_N, S_{N+2} - S_N, \cdots, S_n - S_N) > -l + \beta n^{1/2}\}$$

$$= \sum_{l=-N}^{N} \text{Prob}\, \{S_N = l\} P(m, l - p)$$

where $m = n - N$ and $p = [\beta n^{1/2}]$. Using the previous approximations, we then have

$$(2) \qquad P_n(\beta) = \frac{1}{\pi} \sum_{l=-p}^{N}{}^{*} \left\{ 2N^{-1/2} e^{-l^2/2N} \sum_{j=0}^{l-p}{}^{*} 2m^{-1/2} e^{-j^2/2m} \right\} + \epsilon_2$$

where $|\epsilon_2| < Cn^{-1/2}$, C being bounded independently of n. The sums in (2) correspond to Riemann sums and, letting $n \to \infty$, we have

$$\lim_{n\to\infty} P_n(\beta) = \frac{1}{\pi} \int_{u=\beta\alpha^{-1/2}}^{\infty} e^{-u^2/2} du \int_{t=0}^{(u\alpha^{1/2}-\beta)(1-\alpha)^{-1/2}} e^{-t^2/2} dt = V_1(\beta).$$

The proof is now easily completed. Applying our particular result (1) we have

$$(3) \qquad \begin{aligned} \text{Prob}\, \{\min (\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_k) > (\beta + \epsilon) k^{1/2}\} - 1/k\epsilon^2 &\leqq V_1(\beta) \\ &\leqq \text{Prob}\, \{\min (\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_k) > \beta k^{1/2}\}. \end{aligned}$$

Replacing β by $\beta - \epsilon$, we have

$$\text{Prob}\, \{\min (\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_k) > \beta k^{1/2}\} - 1/k\epsilon^2 \leqq V_1(\beta - \epsilon)$$

and replacing β by $\beta + \epsilon$ in (3), we have

$$V_1(\beta + \epsilon) \leqq \text{Prob}\, \{\min (\overline{R}_1, \cdots, \overline{R}_k) > (\beta + \epsilon) k^{1/2}\}$$

and applying these results to (1), we find that for the general case

$$V_1(\beta + \epsilon) - \frac{1}{k\epsilon^2} \leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \leqq V_1(\beta - \epsilon) + 1/k^2.$$

The proof is completed by letting $k \to \infty$ while ϵ is held fixed and then letting $\epsilon \to 0$ and using the fact that $V_1(\beta)$ is continuous. The corollary follows by calculating $V_1(0)$ explicitly.

3. **Proof of II.** Letting

$$P_n(\beta) = \text{Prob}\left\{\frac{1}{n^2}\sum_{j=1}^{n} p(j/n)S_j^2 < \beta\right\},$$

and again following the method of Erdös and Kac, we find, for any $\epsilon>0$ and all positive integers k, that

$$\begin{aligned}
&\text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} p(j/k)R_j^2 < \beta - \epsilon\right\} - C/\epsilon k^{1/2} \\
&\qquad \leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \\
&\qquad \leqq \text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} P(j/k)R_j^2 < \beta + \epsilon\right\} + C/\epsilon k^{1/2},
\end{aligned} \tag{4}$$

where C is a constant, $R_j = G_1 + G_2 + \cdots + G_j$, and $G_1, G_2, \cdots$ are independent, normally distributed random variables each having mean 0 and standard deviation 1. The remainder of the proof is essentially a calculation of $\lim_{k\to\infty}\phi_k(\xi)$ where $\phi_k(\xi)$ is the characteristic function of the distribution of $(1/k^2)\sum_{j=1}^{k} p(j/k)R_j^2$.

We consider first the case in which $p(t) > 0$, $0 \leqq t \leqq 1$, and for convenience we let $p_j = p(j/k)$. We then have

$$\begin{aligned}
\phi_k(\xi) &= (2\pi)^{-k/2}\int_{-\infty}^{\infty}\!\!\cdots\int \exp\left\{i\xi/k^2\sum_{j=1}^{k} p_j\left(\sum_{i=1}^{j} x_i\right)^2 \right. \\
&\qquad\qquad \left. - \frac{1}{2}\sum_{j=1}^{k} x_j^2\right\} dx_1\cdots dx_k \\
&= (2\pi)^{-k/2}\int_{-\infty}^{\infty}\!\!\cdots\int \exp\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k} p_j y_j^2 - \frac{1}{2}y_1^2 \right. \\
&\qquad\qquad \left. - \frac{1}{2}\sum_{j=2}^{k}(y_j - y_{j-1})^2\right\} dy_1\cdots dy_k.
\end{aligned} \tag{5}$$

Let $A = ((a_{ij}))$ be the matrix of the quadratic form

$$y_1^2 + \sum_{j=2}^{k}(y_j - y_{j-1})^2$$

and let the transformation $z_j = p_j^{1/2} y_j$ take A into $B = ((b_{ij}))$, so that $b_{ij} = (p_i p_j)^{-1/2} a_{ij}$. We then have

$$\begin{aligned}
\phi_k(\xi) = (2\pi)^{-k/2}(p_1 p_2\cdots p_k)^{-1/2}\int_{-\infty}^{\infty}\!\!\cdots\int \exp&\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k} z_j^2 \right. \\
&\left. - \frac{1}{2}\sum_{i,j=1}^{k} b_{ij}z_i z_j\right\} dz_1\cdots dz_k,
\end{aligned}$$

and if we denote the eigenvalues of B by $\mu_{1k}, \mu_{2k}, \cdots, \mu_{kk}$, it follows easily that

$$\phi_k(\xi) = \prod_{j=1}^{k} \left\{ p_j \left(\mu_{jk} - \frac{2i\xi}{k^2} \right) \right\}^{-1/2}.$$

Since $\phi_k(0) = 1$, we have

$$\phi_k(\xi) = \prod_{j=1}^{k} \left\{ 1 - \frac{2i\xi}{k^2 \mu_{jk}} \right\}^{-1/2}. \tag{6}$$

Now $B = C^{-1}$ where $C = ((p_i^{1/2} c_{ij} p_j^{1/2}))$ and $((c_{ij})) = ((\min \{i, j\})) = A^{-1}$. It then follows that $1/\mu_{1k}, 1/\mu_{2k}, \cdots, 1/\mu_{kk}$ are the eigenvalues of C. From Hilbert's approach to Fredholm's theory [8, p. 14], it follows that $1/k^2\mu_{jk} \to \lambda_j$ where $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of the integral equation

$$\int_0^1 (p(s))^{1/2} \min (s, t)(p(t))^{1/2} f(t) dt = \lambda f(s).$$

Letting $g(s) = f(s)(p(s))^{-1/2}$, we obtain

$$\int_0^1 \min (s, t) p(t) g(t) dt = \lambda g(s)$$

and differentiating twice we find that

$$\lambda g''(s) + p(s) g(s) = 0. \tag{7}$$

Thus $\lambda_1, \lambda_2, \cdots$ are the eigenvalues of (7), subject to the conditions

$$g(0) = g'(1) = 0$$

which arise from the integral equation. Since $\min (s, t)$ is a positive definite kernel, it follows that $(p(s))^{1/2} \min (s, t)(p(t))^{1/2}$ is positive definite and by Mercer's theorem we have $\lambda_j > 0$ for each j and also $\sum_{j=1}^{\infty} \lambda_j$ converges. Hence we may pass to the limit in (6) and we have

$$\lim_{k \to \infty} \phi_k(\xi) = \prod_{j=1}^{\infty} (1 - 2i\xi\lambda_j)^{-1/2}, \tag{8}$$

the convergence being uniform in every finite ξ-interval.

We now remove the restriction that $p(t)$ be positive and we require that $p(t)$ have a most a denumerable set of zeros in the unit interval. Moreover, using a continuity argument, it is easily seen that we need only consider the case where the zeros of $p(t)$ are irrational. Letting $u = 2i\xi/k^2$, it follows from (5) that

$$\phi_k(\xi) = (2\pi)^{-k/2}\int_{-\infty}^{\infty}\cdots\int \exp\left\{-\frac{1}{2}\left(\sum_{i,j=1}^{k} a_{ij}y_iy_j - u\sum_{j=1}^{k} p_jy_j^2\right)\right\}dy_1\cdots dy_k.$$

Letting

$$J = \begin{bmatrix} a_{11}-up_1 & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22}-up_2 & \cdots & a_{2k} \\ \vdots & & \ddots & \\ a_{k1} & & \cdots & a_{kk}-up_k \end{bmatrix}$$

it follows that

$$\phi_k(\xi) = |J|^{-1/2}.$$

Letting $H=((h_{ij}))$ where $h_{ij}=a_{ij}/p_i$, we have

$$|J| = p_1p_2\cdots p_k|H-uI|,$$

where I is the identity matrix. Moreover, if we denote the eigenvalues of $\lambda_{1k}, \lambda_{2k}, \cdots, \lambda_{kk}$, it follows that the eigenvalues of $H-uI$ are $\lambda_{1k}-u\lambda_{2k}-u, \cdots, \lambda_{kk}-u$. It then follows easily that

$$\phi_k(\xi) = \prod_{j=1}^{k}\{p_j(\lambda_{jk}-u)\}^{-1/2}.$$

Using the fact that $\phi_k(0)=1$, we have

$$\phi_k(\xi) = \prod_{j=1}^{k}\left(1-\frac{2i\xi}{k^2\lambda_{jk}}\right)^{-1/2}. \tag{9}$$

Moreover, it is clear that $1/\lambda_{1k}, 1/\lambda_{2k}, \cdots, 1/\lambda_{kk}$ are the eigenvalues of the matrix $G=((g_{ij}))=H^{-1}$ where

$$g_{ij} = p_j\min(i,j) = p(j/k)\min(i,j).$$

To evaluate $\lim_{k\to\infty}\phi_k(\xi)$ we again make use of Hilbert's considerations. We shall be concerned with the integral equation

$$f(s)-\lambda\int_0^1 \min(s,t)p(t)f(t)dt = 0 \tag{10}$$

and for convenience we write $K(s,t)=\min(s,t)\,p(t)$. Now let $K_{ij}=K(i/k,j/k)=\min(i,j)p_j/k$. Following Hilbert [8, p. 9] we define $D_k(l)$ to be the determinant

$$\begin{vmatrix} 1 - lK_{11} & -lK_{12} \cdots & -lK_{1k} \\ -lK_{21} & 1 - lK_{22} \cdots & -lK_{2k} \\ \vdots & & \\ -lK_{k1} & -lK_{k2} \cdots & 1 - lK_{kk} \end{vmatrix}$$

and it follows that

$$\lim_{k\to\infty} D_k(\lambda/k) = D(\lambda) \tag{11}$$

where $D(\lambda)$ is the Fredholm determinant associated with (10) and is defined over the entire complex λ-plane. Moreover, the convergence is uniform in every bounded region of the complex plane. It is easily seen that

$$D_k(l) = \left| J - \frac{l}{k} G \right|$$

and since the eigenvalues of G are $1/\lambda_{1k}, 1/\lambda_{2k}, \cdots, 1/\lambda_{kk}$, it easily follows that

$$D_k(l) = \prod_{j=1}^{k} \left(1 - \frac{l}{k\lambda_{jk}}\right),$$

and from (9) we have $\phi_k(\xi) = (D_k(2i\xi/k))^{-1/2}$. Letting $k \to \infty$ and using (11) we have

$$\lim_{k\to\infty} \phi_k(\xi) = (D(2i\xi))^{-1/2}. \tag{12}$$

Since $|\phi_k(\xi)| \leqq 1$ for all k and for all real ξ, it follows that $D(2i\xi) \neq 0$ and $\lim_{k\to\infty} \phi_k(\xi)$ is defined for all real ξ. Furthermore, it is easily seen that (12) is equivalent to (8) when $p(t) > 0$. In the general case it follows readily that (10) is equivalent to (7) and it is known that (7) will have eigenvalues even if we remove the restriction that $p(t)$ be positive. However, in the general case we are unable to pass to the limit termwise as in (8) since our integral equation would not have a positive definite kernel. Therefore, we must resort to the Fredholm determinant.

As yet we have not shown the existence of the limiting distribution in the general case. We first consider the special case of normally distributed random variables. We have shown that $\phi_k(\xi)$ approaches a certain function $\phi(\xi)$ uniformly in every finite ξ-interval as k becomes infinite. Applying the continuity theorem for Fourier-Stieltjes transforms, we see that there exists a distribution function $V_2(\beta)$ such that

$$\lim_{k\to\infty} \text{Prob} \left\{ \frac{1}{k^2} \sum_{j=1}^{k} p(j/k) R_j^2 < \beta \right\} = V_2(\beta)$$

at every continuity point of $V_2(\beta)$. Furthermore, we have

$$\phi(\xi) = \int_{-\infty}^{\infty} e^{i\beta\xi} dV_2(\beta).$$

For $p(t) > 0$, we have

$$|\phi(\xi)| = \prod_{j=1}^{\infty} |1 - 2i\xi\lambda_j|^{-1/2} \leqq \prod_{j=1}^{N} |1 - 2i\xi\lambda_j|^{-1/2}$$

since $|1-2i\xi\lambda_j| \geqq 1$. Taking $N \geqq 4$, we see that $|\phi(\xi)|$ is integrable over $(-\infty, \infty)$ and it is evident from the inversion formula for Fourier transforms that $V_2(\beta)$ must be continuous at all points. The theorem then follows from (4). In the general case, however, we are unable to show that $V_2(\beta)$ is continuous everywhere and the theorem holds only at the continuity points of $V_2(\beta)$.

4. **Proof of III.** Letting

$$P_n(\beta) = \text{Prob}\left\{\frac{1}{n^2}\sum_{j=1}^{n} S_{nj}^2 < \beta\right\},$$

we find, for any $\epsilon > 0$ and for all positive integers k, that

$$\begin{aligned} &\text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} R_{kj}^2 < \beta - \epsilon\right\} - C/\epsilon k^{1/2} \\ &\qquad \leqq \liminf_{n\to\infty} P_n(\beta) \leqq \limsup_{n\to\infty} P_n(\beta) \\ &\qquad \leqq \text{Prob}\left\{\frac{1}{k^2}\sum_{j=1}^{k} R_{jk}^2 < \beta + \epsilon\right\} + C/\epsilon k^{1/2} \end{aligned} \tag{13}$$

where C is a constant, $R_{kj} = G_{k1} + G_{k2} + \cdots + G_{kj}$, and $G_{k1}, \cdots, G_{kj}$ are independent, normally distributed random variables each having mean $\mu k^{-1/2}$ and standard deviation 1. We now compute $\lim_{k\to\infty} \phi_k(\xi)$ where $\phi_k(\xi)$ is the characteristic function of the distribution of

$$\frac{1}{k^2}\sum_{j=1}^{k} R_{kj}^2.$$

We then have

$$\begin{aligned} \phi_k(\xi) = (2\pi)^{-k/2} \int_{-\infty}^{\infty}\cdots\int \exp\left\{\frac{i\xi}{k^2}\sum_{j=1}^{k}\left(\sum_{i=1}^{j} x_i\right)^2\right\} \\ \cdot\exp\left\{-\frac{1}{2}\sum_{j=1}^{k}(x_j - \mu k^{-1/2})^2\right\} dx_1 \cdots dx_k, \end{aligned}$$

and making the substitution $y_j = \sum_{i=1}^{j} x_i$ we have

$$\phi_k(\xi) = (2\pi)^{-k/2} e^{-\mu^2/2} \int_{-\infty}^{\infty} \cdots \int \exp \left\{ \frac{\mu y_k}{k^{1/2}} + \frac{i\xi}{k^2} \sum_{j=1}^{k} y_j^2 \right\} \cdot \exp \left\{ -\frac{1}{2} y_1^2 - \frac{1}{2} \sum_{j=2}^{k} (y_j - y_{j-1})^2 \right\} dy_1 \cdots dy_k.$$

We shall make use of matrices again and we shall denote the transpose of a matrix by a superscript "T." Let

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_k \end{bmatrix} \quad \text{and} \quad t = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ \mu k^{1/2} \end{bmatrix},$$

y and t being column matrices each with k rows. Furthermore, let A be the $k \times k$ matrix of the quadratic form

$$y_1^2 + \sum_{j=2}^{k} (y_j - y_{j-1})^2.$$

It then follows that

$$\phi_k(\xi) = (2\pi)^{-k/2} e^{-\mu^2/2} \int_{-\infty}^{\infty} \cdots \int \cdot \exp \left\{ \frac{i\xi}{k^2} y^T y - \frac{1}{2} y^T A y + t^T y \right\} dy_1 \cdots dy_k. \tag{14}$$

Let A have eigenvalues $\lambda_{1k}, \lambda_{2k}, \cdots, \lambda_{kk}$ and let C be an orthogonal matrix such that

$$C^T A C = D,$$

where D is the diagonal matrix

$$\begin{bmatrix} \lambda_{1k} & & & 0 \\ & \lambda_{2k} & & \\ & & \ddots & \\ 0 & & & \lambda_{kk} \end{bmatrix}.$$

Let

$$z = \begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_k \end{bmatrix},$$

and applying the transformation $y=Cz$ to (14) we have

$$\phi_k(\xi) = (2\pi)^{-k/2}e^{-\mu^2/2}\int\overset{\infty}{\underset{-\infty}{\cdots}}\int \cdot\exp\left\{\frac{i\xi}{k^2}z^Tz - \frac{1}{2}z^TDz + u^Tz\right\}dz_1\cdots dz_k,$$

where

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_k \end{bmatrix} = C^Tt.$$

We may now separate variables in the last integral and it follows easily that

$$\phi_k(\xi) = \left\{\prod_{j=1}^k\left(\lambda_{jk} - \frac{2i\xi}{k^2}\right)^{-1/2}\right\} \cdot\exp\left\{-\frac{\mu^2}{2} + \frac{1}{2}\sum_{j=1}^k \frac{u_j^2}{\lambda_{jk} - 2i\xi/k^2}\right\}.$$

We define $Z^{1/2}$ in the plane with the negative real axis removed and satisfying the condition that $Z^{1/2}$ be positive for real and positive Z. Since $\phi_k(0)=1$ for any μ, we consider the case $\mu=0$ and we find that $\lambda_{1k}\lambda_{2k}\cdots\lambda_{kk}=1$. We then have

$$\phi_k(\xi) = \left\{\prod_{j=1}^k\left(1 - \frac{2i\xi}{\lambda_{jk}k^2}\right)^{-1/2}\right\}\exp\left\{-\frac{\mu^2}{2} + \frac{1}{2}\sum_{j=1}^k \frac{u_j^2}{\lambda_{jk} - 2i\xi/k^2}\right\}.$$

Let

$$q = \sum_{j=1}^k \frac{u_j^2}{\lambda_{jk} + \alpha}$$

where

$$\alpha = -2i\xi/k^2$$

and it readily follows that

$$q = t^T(D + \alpha I)^{-1}t$$

where I is the identity matrix. We then find that

$$q = t^T(A + \alpha I)^{-1}t$$

and letting

$$B = (A + \alpha I)^{-1} = ((b_{ij}))$$

we find that

$$q = \frac{u^2}{k} b_{kk}.$$

We then have

$$(15) \qquad \phi_k(\xi) = \left\{ \prod_{j=1}^{k} \left(1 - \frac{2i\xi}{\lambda_{jk}k^2}\right)^{-1/2} \exp\left\{\frac{\mu^2}{2}\left(\frac{b_{kk}}{k} - 1\right)\right\}\right\}.$$

We wish to let $k \to \infty$ in (15). It is known [1, p. 299] that

$$(16) \qquad \lim_{k\to\infty} \prod_{j=1}^{k} \left(1 - \frac{2i\xi}{\lambda_{jk}k^2}\right)^{-1/2} = (\operatorname{sech}(-2i\xi)^{1/2})^{1/2}$$

and the convergence is uniform in every finite ξ-interval. Hence we need only consider $\lim_{k\to\infty} b_{kk}/k$. Let $F = A + \alpha I = ((f_{ij}))$ where $\alpha = -2i\xi/k^2$ and it readily follows that

$$b_{kk} = |F_{kk}| / |F|$$

where $|F|$ is the determinant of F and $|F_{kk}|$ is the co-factor of f_{kk}. To evaluate these determinants we make use of the following well known result:

Let

$$D_m(a) = \begin{vmatrix} 1 & a & & 0 \\ a & 1 & \ddots & \\ & \ddots & \ddots & a \\ 0 & & a & 1 \end{vmatrix}$$

where m is the order of the determinant. If $a^2 \neq 1/4$, we have

$$D_m(a) = \frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2}$$

where r_1 and r_2 are the roots of $x^2 - x + a^2 = 0$.

We shall be interested in $D_m(a)$ as $m \to \infty$ and $a^2 \to 1/4$ and we shall find it more convenient to use $G_m(a)$ where

$$G_m(a) = 2^m \frac{r_1^{m-1} - r_2^{m-1}}{r_1 - r_2} = 2^m D_m(a).$$

Since F is the $k \times k$ matrix

$$\begin{bmatrix} 2+\alpha & -1 & & & 0 \\ -1 & 2+\alpha & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 2+\alpha & -1 \\ 0 & & & -1 & 1+\alpha \end{bmatrix}$$

it follows that

$$F = a_2^k \begin{vmatrix} 1 & a & & 0 \\ a & \ddots & \ddots & \\ & \ddots & 1 & a \\ 0 & & a & a_1/a_2 \end{vmatrix} \tag{17}$$

where

$$a_1 = 1 + \alpha, \qquad a_2 = 2 + \alpha, \qquad a = -1/a_2.$$

Expanding the determinant in (17) by the last row, we find that

$$|F| = \frac{a_2^{k-2}}{2^{k-1}} (a_1 a_2 G_{k-1}(a) - 2G_{k-2}(a)).$$

Similarly we have

$$|F_{kk}| = \frac{a_2^{k-1}}{2^{k-1}} G_{k-1}(a),$$

and combining the last two equations we have

$$b_{kk} = \frac{a_2 G_{k-1}(a)}{a_1 a_2 G_{k-1}(a) - 2G_{k-2}(a)}. \tag{18}$$

We next obtain estimates on $G_{k-1}(a)$ and $G_{k-2}(a)$ for large k and we shall use C to denote a number bounded independently of k. The various C's will not necessarily be equal. Since r_1 and r_2 are the roots of $x^2 - x + a^2 = 0$, we write

$$r_1 = (1 + \epsilon)/2$$

and

$$r_2 = (1 - \epsilon)/2$$

where

$$\epsilon = (1 - 4a^2)^{1/2}.$$

It is easily seen that

$$\epsilon = \frac{(2\xi_1)^{1/2}}{k} + \frac{C}{k^3} \tag{19}$$

where $\xi_1 = -i\xi$. Now

$$G_{k-1}(a) = 2^{k-1}\left(\frac{r_1^k - r_2^k}{r_1 - r_2}\right) = \sum_{j=1}^{k} (1 + \epsilon)^{j-1}(1 - \epsilon)^{k-j}.$$

Letting

$$Z_j = (1 + \epsilon)^{j-1}(1 - \epsilon)^{k-j},$$

it follows easily from (19) that

$$\log Z_j = -(2\xi_1)^{1/2} - \frac{1}{k}(\xi_1 + (2\xi_1)^{1/2}) + \frac{2j(2\xi_1)^{1/2}}{k} + C/k^2$$

and

$$Z_j = C/k^2 + \exp\left\{-(2\xi_1)^{1/2} - \frac{1}{k}(\xi_1 + (2\xi_1)^{1/2}) + \frac{2j}{k}(2\xi_1)^{1/2}\right\}.$$

Finally, we find that

$$G_{k-1}(a) = (k - \xi_1)\frac{\sinh (2\xi_1)^{1/2}}{(2\xi_1)^{1/2}} + \frac{C}{k}. \tag{20}$$

Since

$$G_{k-2}(a) = \sum_{j=1}^{k-1}(1 + \epsilon)^{j-1}(1 - \epsilon)^{k-j-1}$$

we have

$$G_{k-2}(a) = \frac{1}{1 - \epsilon}\{G_{k-1}(a) - (1 + \epsilon)^{k-1}\}.$$

It follows readily from (20) that

$$G_{k-2}(a) = (k - \xi_1)\frac{\sinh (2\xi_1)^{1/2}}{(2\xi_1)^{1/2}} - \cosh (2\xi_1)^{1/2} + \frac{C}{k}. \tag{21}$$

Combining (18), (20), and (21), we find

$$\frac{b_{kk}}{k} = \frac{2(k - \xi_1)\sinh (2\xi_1)^{1/2}/(2\xi_1)^{1/2} + C/k}{k(2\cosh (2\xi_1)^{1/2} + C/k)}$$

and letting $k \to \infty$, we have

$$\lim_{k\to\infty}\frac{b_{kk}}{k} = \frac{\tanh (2\xi_1)^{1/2}}{(2\xi_1)^{1/2}}.$$

Letting $k \to \infty$ in (15) and using (16), we find that

$$\begin{aligned}\lim_{k\to\infty}\phi_k(\xi) &= (\operatorname{sech}(-2i\xi)^{1/2})^{1/2} \\ &\quad\cdot\exp\left\{\frac{u^2}{2}\left(\frac{\tanh(-2i\xi)^{1/2}}{(-2i\xi)^{1/2}} - 1\right)\right\}\end{aligned} \tag{22}$$

and the convergence is uniform in every finite ξ-interval.

The proof is completed as in the previous section by applying the continuity theorem for Fourier-Stieltjes transforms. Thus there exists a distribution function $V_3(\beta)$ such that

$$\lim_{k\to\infty} \text{Prob} \left\{ \frac{1}{k^2} \sum_{j=1}^{k} R_{kj}^2 < \beta \right\} = V_3(\beta)$$

at every continuity point of $V_3(\beta)$. Furthermore, if we let

$$\phi(\xi) = \lim_{k\to\infty} \phi_k(\xi)$$

we have

$$\phi(\xi) = \int_{-\infty}^{\infty} e^{i\beta\xi} dV_3(\beta).$$

It is easily seen from (22) that $\phi(\xi)$ approaches zero exponentially as $|\xi|$ becomes infinite and therefore $|\phi(\xi)|$ is integrable over $(-\infty, \infty)$. Hence $V_3(\beta)$ must be continuous at all points and the theorem follows easily from (13).

BIBLIOGRAPHY

1. P. Erdös and M. Kac, *On certain limit theorems of the theory of probability*, Bull Amer. Math. Soc. vol. 52 (1946) pp. 292–302.

2. ———, *On the number of positive sums of independent random variables*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1011–1029.

3. P. Lévy, *Un théorème d'invariance projective relatif au mouvement Brownien*, Comment. Math. Helv. vol. 16 (1943–1944) pp. 242–248.

4. M. Kac and A. J. F. Siegert, *On the theory of noise in radio receivers with square law detectors*, Journal of Applied Physics vol. 12 (1947) pp. 383–397.

5. R. H. Cameron and W. T. Martin, *The Wiener measure of Hilbert neighborhoods in the space of real continuous functions*, Journal of Mathematics and Physics vol. 23 (1944) pp. 195–209.

6. ———, *Evaluation of various Wiener integrals by use of certain Sturm-Liouville differential equations*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 73–89.

7. J. V. Uspensky, *Introduction tp mathematical probability*, New York, 1937.

8. D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Berlin and Leipzig, Teubner, 1912.

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