CONVEXITY THEOREMS

R. SALEM

In spite of the generality of our title, we do not intend to give here a survey of all convexity theorems. Most of them, like the three circles theorem of Hadamard, are too classical to be commented on here. We shall confine ourselves to Marcel Riesz's convexity theorem, which is one of the very powerful tools of modern analysis, and to certain of its recent extensions.

1. **Marcel Riesz's theorem.** Let

$$f = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j$$

be a bilinear form, where the constants a_{ij} are real or complex, and the variables x_i, y_j are essentially supposed to be complex. Let $\alpha \geqq 0$, $\beta \geqq 0$ be real numbers, and denote by $M(\alpha, \beta)$ the maximum of $|f|$ under the conditions

$$\sum_1^m | x_i |^{1/\alpha} \leqq 1, \qquad \sum_1^n | y_j |^{1/\beta} \leqq 1.$$

(If $\alpha = 0$ the first condition means that $|x_i| \leqq 1$ for $i = 1, 2, \cdots, m$, and the same remark applies to the second condition.) Then $\log M(\alpha, \beta)$ is a convex function of α, β in the quadrant $\alpha \geqq 0, \beta \geqq 0$; in other words if $0 < t < 1$ and if

$$\alpha = \alpha_1 t + \alpha_2(1 - t), \qquad \beta = \beta_1 t + \beta_2(1 - t),$$

then

$$M(\alpha, \beta) \leqq M^t(\alpha_1, \beta_1) M^{1-t}(\alpha_2, \beta_2).$$

This is M. Riesz's fundamental theorem. (See M. Riesz [5][1] and a different proof in Paley [4]; see also a generalization of the theorem in L. C. Young [11].) M. Riesz's argument proved the convexity only in the triangle $0 \leqq \alpha \leqq 1$, $0 \leqq \beta \leqq 1$, $\alpha + \beta \geqq 1$. The extension to the whole quadrant is due to Thorin [9]. We shall not give the proof of the theorem here, since we intend to sketch later on the proof of a more general result. Let us only point out that if we restrict the

An address delivered before the New York Meeting of the Society on October 30, 1948, by invitation of the Committee to Select Hour Speakers for Eastern Sectional Meetings; received by the editors November 23, 1948.

[1] Numbers in brackets refer to the references cited at the end of the paper.

variables to be real, the convexity theorem can not be extended beyond the domain $\alpha+\beta\geqq 1$, $\alpha\geqq 0$, $\beta\geqq 0$. For the proof of this result, we refer the reader to Thorin [10]. We add that in the classical applications of the theorem the convexity in Riesz's triangular domain is sufficient.

In its applications, M. Riesz's theorem is mostly used in a different form. Let us write

$$X_j(x) = \sum_{i=1}^{m} a_{ij}x_i \qquad (j = 1, 2, \cdots, n).$$

Let now $0\leqq\gamma\leqq 1$ and $\alpha\geqq 0$, and denote by $\mathfrak{M}(\alpha, \gamma)$ the maximum of

$$\left(\sum_1^n |X_j|^{1/\gamma}\right)^{\gamma}$$

when

$$\sum_1^m |x_i|^{1/\alpha} \leqq 1.$$

It is a classical consequence of Hölder's theorems that, if $\beta=1-\gamma$ and $M(\alpha, \beta)$ has the same meaning as above,

$$\mathfrak{M}(\alpha, \gamma) = \max \frac{\left(\sum_1^n |X_j|^{1/\gamma}\right)^{\gamma}}{\left(\sum_1^m |x_i|^{1/\alpha}\right)^{\alpha}} = M(\alpha, \beta)$$

and thus $\mathfrak{M}(\alpha, \gamma)$ is convex in the domain $\alpha\geqq 0$, $0\leqq\gamma\leqq 1$. (If the variables are real, the theorem is true in the domain $\alpha\geqq 0$, $0\leqq\gamma\leqq 1$, $\gamma\leqq\alpha$.)

Finally, it is hardly worth mentioning that the theorem remains valid if we consider, under the same conditions for the exponents, the maximum of

$$\frac{\left(\sum_1^n \sigma_j |X_j|^{1/\gamma}\right)^{\gamma}}{\left(\sum_1^m \rho_i |x_i|^{1/\alpha}\right)^{\alpha}}$$

where $\rho_i>0$, $\sigma_j>0$.

2. **The applications of M. Riesz's theorem.** The most general application of M. Riesz's theorem, which is due to M. Riesz himself,

can be stated as follows.

Let us denote generally by $L^{r,\phi}(t_1, t_2)$, for $r \geqq 1$, the class of (complex-valued) functions $f(t)$ such that the Lebesgue-Stieltjes integral $\int_{t_1}^{t_2} |f|^r d\phi$, taken with respect to the nondecreasing function $\phi(t)$, is finite. By $\mathfrak{N}_{r,\phi}[f]$ we shall denote the norm $(\int_{t_1}^{t_2} |f|^r d\phi)^{1/r}$. Let $\phi(t)$ $(t_1 \leqq t \leqq t_2)$ and $\psi(u)$ $(u_1 \leqq u \leqq u_2)$ be two nondecreasing functions. Let $a \geqq 1$, $c \geqq 1$ and suppose that there exists a linear operation T associating to every $f(t) \in L^{a,\phi}(t_1, t_2)$ a function $T(f) = g(u) \in L^{c,\psi}(u_1, u_2)$ such that the ratio

$$\frac{\mathfrak{N}_{c,\psi}[g]}{\mathfrak{N}_{a,\phi}[f]}$$

is bounded independently of f. The operation T will be called of the type (a, c). The least value of this ratio, that is, the modulus of the operation, will be denoted by $M(\alpha, \gamma)$, where $\alpha = 1/a$, $\gamma = 1/c$.

Now let ϕ and ψ be fixed, and let T be an operation which is simultaneously of the types (a_1, c_1) and (a_2, c_2). Write $\alpha_i = 1/a_i$, $\gamma_i = 1/c_i$. Then, given any point (α, γ) on the segment joining $P_1(\alpha_1, \gamma_1)$ and $P_2(\alpha_2, \gamma_2)$, the operation T can be extended so as to become also of the type (a, c) where $a = 1/\alpha$, $c = 1/\gamma$. Moreover $\log M(\alpha, \gamma)$ is convex on the segment P_1P_2. (The domain of convexity is to be reduced to the triangle $0 \leqq \alpha \leqq 1$, $0 \leqq \gamma \leqq \alpha$ if the functions are real-valued.)

For the proof we refer the reader to M. Riesz's original memoir [5] or to Zygmund [13], whose convenient notations we have adopted here. Let us only point out that the result is essentially a consequence (1°) of M. Riesz's theorem; (2°) of the fact that the operation T is linear; (3°) of the fact that since the set of step functions is everywhere dense in the class $L^{r,\phi}$ $(1 \leqq r < \infty)$, we can approximate the functions of the class by step functions, and thus consider, instead of integrals, finite sums.

The most familiar consequence of this result is the Young-Hausdorff theorem, in the general form given by F. Riesz. Let $\phi_1, \phi_2, \cdots, \phi_n, \cdots$ be an orthonormal set of complex-valued functions in (a, b), uniformly bounded, $|\phi_n| \leqq M$, and let $\int_a^b f\bar{\phi}_n dt$ be the Fourier coefficient of the complex-valued function $f(t)$ with respect to ϕ_n. If we are given any sequence of numbers $\{c_n\}$ such that $\sum |c_n|^2 < \infty$, we know that there exists an $f(t) \in L^2$ having the c_n's as Fourier coefficients; moreover $\int_a^b |f|^2 dt = \sum |c_n|^2$. If, in addition, $\sum |c_n| < \infty$, f having the same meaning, one has $|f| \leqq M \sum |c_n|$.

$f(t)$ is obtained from $\{c_n\}$ by a linear operation of the type $(2, 2)$ with modulus 1, and of the type $(1, \infty)$ with modulus M. Hence, if

$$\begin{aligned} \alpha &= \lambda/2 + 1\cdot(1-\lambda) = 1 - \lambda/2, \\ \beta &= \lambda/2 + 0\cdot(1-\lambda) = \lambda/2, \end{aligned} \qquad (0 < \lambda < 1)$$

the operation is also of the type $(2/(2-\lambda), 2/\lambda)$ with modulus $M^{1-\lambda}$. Setting $2/(2-\lambda) = p$, $2/\lambda = p'$, one has

$$1 < p < 2, \; 1/p + 1/p' = 1, \; 1 - \lambda = 2/p - 1$$

and

$$\left(\int_a^b |f|^{p'} dt\right)^{1/p'} \leqq M^{(2-p)/p}\left(\sum |c_n|^p\right)^{1/p}.$$

The proof of the other part of the Young-Hausdorff theorem is similar.

The Young-Hausdorff theorem is valid only if the orthogonal system is uniformly bounded. A generalization of the theorem in the case of an unbounded system has been given by Marcinkiewicz and Zygmund [**3**]. Assuming that the functions ϕ_n satisfy the inequality

$$\left(\int_a^b |\phi_n|^{\nu} dt\right)^{1/\nu} \leqq M_n$$

for a certain $\nu > 2$, and applying in the same way as above M. Riesz's theorem, the authors get, instead of the preceding inequality

$$\left(\int_a^b |f|^{p'} dt\right)^{1/p'} \leqq \left(\sum_1^\infty M_n^{2-p} |c_n|^p\right)^{1/p} \quad (1 \leqq p \leqq 2)$$

but then p' is given by the equation

$$\frac{2-\mu}{p} + \frac{\mu}{p'} = 1$$

where $1/\mu + 1/\nu = 1$; besides, the series $\sum |c_n|^2$ is supposed to be convergent. (This condition is necessary only if the interval (a, b) is infinite.) The second part of the Young-Hausdorff theorem is generalized in a similar fashion.

Let us finally quote the following theorem, concerning Rademacher's functions and due to Zygmund [**14**]. Starting from the inequality

$$\left(\int_0^1 |f(t)|^k dt\right)^{1/k} \leqq k^{1/2}\left(\sum_0^\infty |c_n|^2\right)^{1/2}$$

where $f(t) = \sum_0^\infty c_n\phi_n(t)$, $\phi_n(t)$ being the nth Radamacher function, an inequality which holds whenever $\sum |c_n|^2 < \infty$ and $k \geqq 2$, and using the relation

$$|f(t)| \leqq \sum_0^\infty |c_n|,$$

the author gets

$$\left(\int_0^1 |f(t)|^k dt\right)^{1/k} \leqq k^{1/r'}\left(\sum_0^\infty |c_n|^r\right)^{1/r}$$

whenever $\sum |c_n|^r < \infty$, $1 \leqq r \leqq 2$, $k \geqq 1$, and r' is the complementary exponent to r $(1/r+1/r'=1)$.

3. **Thorin's generalization of M. Riesz's theorem.** Thorin's first generalization [9] not only allows the extension, which we have already mentioned, of M. Riesz's triangular domain of convexity to the whole quadrant, but, what is much more important for new applications, it shows that instead of the maximum of a bilinear form, we can consider the maximum of any entire function of n complex variables.

The theorem of Thorin can be stated as follows: Let $f(z_1, z_2, \cdots, z_n)$ be an entire function of the n complex variables $z_1, z_2, \cdots, z_n$. Let V be a bounded domain in the n-dimensional euclidean space with coordinates $v_1, v_2, \cdots, v_n$. Let us denote by $\alpha_1, \alpha_2, \cdots, \alpha_n$ nonnegative exponents, and by $M(\alpha_1, \alpha_2, \cdots, \alpha_n)$ the maximum of $|f(z_1, \cdots, z_n)|$ under the conditions

$$|z_1| = v_1^{\alpha_1}, \cdots, |z_n| = v_n^{\alpha_n} \qquad (v_1, v_2, \cdots, v_n) \in V.$$

Then $\log M(\alpha_1, \cdots, \alpha_n)$ is convex in the domain $\alpha_k \geqq 0$ $(k=1, 2, \cdots, n)$.

The original proof of Thorin [9] was rather long. In 1944, Tamarkin and Zygmund [8] proved in a very elegant way that Thorin's theorem was a simple consequence of the maximum modulus theorem. We shall sketch here a short proof of Thorin's result which has been published recently [6].

Suppose first that $0 < A \leqq v_k \leqq B < \infty$ $(k=1, \cdots, n)$ and let us write $z_k = e^{\alpha_k \xi_k + i\eta_k}$ $(k=1, \cdots, n)$ where the η_k are arbitrary and the point $(\xi_1 \cdots \xi_n)$ belongs to the bounded domain D corresponding to $(e^{\xi_1}, \cdots, e^{\xi_n}) \in V$. Let $\alpha_k = a_k + \lambda_k \log t$ where the a_k and λ_k are real and fixed, and t is a positive real variable. $M(\alpha_1, \cdots, \alpha_n)$ becomes a function of t, $M(t)$, and we have to show that $\log M(t)$ is a convex function of $\log t$. Write, for $p \geqq 1$,

$$I_p(t) = \int \left| f(e^{\xi_1(a_1+\lambda_1 \log t)+i\eta_1}, \cdots, e^{\xi_n(a_n+\lambda_n \log t)+i\eta_n}) \right|^p \cdot d\xi_1 \cdots d\xi_n d\eta_1 \cdots d\eta_n,$$

the integral being extended to the domain

$$0 \leqq \eta_k \leqq 2\pi, \qquad k = 1, \cdots, n; \ (\xi_1 \cdots \xi_n) \in D.$$

If we consider now t as a complex variable, $t = \rho e^{i\sigma}$, $\rho > 0$, the integral is a logarithmically subharmonic function of t, and so $I_p(t)$ is, for a determined choice of the logarithm, a subharmonic logarithmically function of t. Moreover this function is uniform, and $I_p(t) = I_p(\rho)$, thus depending only on the modulus of t. For:

$$\xi_k(a_k + \lambda_k \log \rho + \lambda_k i\sigma) + i\eta_k = \xi_k(a_k + \lambda_k \log \rho) + i(\eta_k + \xi_k\lambda_k\sigma)$$

and our assertion is justified by the fact that the integral is extended to all possible values of η_k, mod 2π. Now $\log I_p(\rho)$ being subharmonic, $\log I_p(\rho)$ is a convex function of $\log \rho$; and since $M(t) = \lim_{p=\infty} [I_p(t)]^{1/p}$ it is sufficient to let p increase infinitely to obtain the asserted result.

The argument remains valid without change if, instead of supposing that $f(z_1, \cdots, z_n)$ is analytic, we assume only that $\log |f|$ is subharmonic, that is, that $|f(z_1 \cdots z_n)|$ is logarithmically subharmonic (in the paper quoted in [6] the word "logarithmically" has been omitted and should be restored). According to Lelong (Ann. École Norm. vols. 61–62 (1945) pp. 301–338) $g(z_1 \cdots z_n)$ is subharmonic if $g[\phi_1(t) \cdots \phi_n(t)]$ is subharmonic for analytic functions $\phi_1(t) \cdots \phi_n(t)$.

As Thorin has pointed out, the convexity holds if we suppose only $v_k \geqq 0$ $(k=1, \cdots, n)$ provided we restrict the domain of convexity from $-\infty < \alpha_k < \infty$ to $\alpha_k \geqq 0$ $(k=1, \cdots, n)$.

In his recent thesis [10] Thorin has given further extensions of the theorem, especially to analytic and subharmonic functionals. We refer the reader to Thorin's paper for these extensions which would require too much space to be quoted here with accuracy.

4. **Applications of Thorin's generalization.** In order to understand the most interesting applications of Thorin's generalized convexity theorem, let us recall the following notations. A function $f(z)$ of the complez variable z, regular for $|z| < 1$, is said to belong to H^p $(p > 0)$ if $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is bounded as $r \to 1$. One writes H instead of H^1. A series (Σ) of the form $\sum_0^\infty c_n e^{ni\theta}$, which is formally $S + iT$, where S and T are conjugate trigonometric series, is said to belong to H^p if $f(z) = \sum_0^\infty c_n z^n \in H^p$. If S (or T) is the Fourier series of a function of

the class L^p ($p>1$), then T(or S) belongs also to L^p and $\Sigma \in H^p$; this, however, is false for $p=1$. If $\Sigma \in H^p$ ($p \geqq 1$), both S and T are Fourier series of the class L^p. In other words the properties H^p for Σ and L^p for S are equivalent for $p>1$ but not for $p=1$.

Now, certain theorems are true for power series of the class H but not for Fourier series of the class L. As an important example, we quote the following theorem due to Zygmund [12]. If $S_n(\theta)$ is the partial sum of order n of the series $f(e^{i\theta}) \sim \sum_0^\infty c_\nu e^{i\nu\theta} \in H$, then

$$\int_0^{2\pi} \frac{| S_{n(\theta)}(\theta) |}{\log [n(\theta) + 2]} d\theta \leqq C_1 \int_0^{2\pi} | f(e^{i\theta}) | d\theta$$

the constant C_1 being independent of f, of the (measurable) function $n(\theta)$ ($0 \leqq n(\theta) \leqq N$), and of N. No such theorem is true for Fourier series of the class L. On the other hand, if $f \in H^2$, one has [1]

$$\int_0^{2\pi} \frac{| S_{n(\theta)}(\theta) |^2}{\log [n(\theta) + 2]} d\theta \leqq C_2 \int_0^{2\pi} | f(e^{i\theta}) |^2 d\theta.$$

If we could apply a convexity theorem here, we would deduce

$$\int_0^{2\pi} \frac{| S_{n(\theta)}(\theta) |^p}{\log [n(\theta) + 2]} \leqq C_p \int_0^{2\pi} | f(e^{i\theta}) |^p d\theta$$

for $f \in H^p$ and $1 \leqq p \leqq 2$, C_p depending on p only; and the theorem would be also true, by what we have said, for Fourier series of the class L^p, but only for $1 < p \leqq 2$. Such a theorem is indeed true; in an important paper [2] Littlewood and Paley have given an extremely difficult proof of it. Hence it is of interest to prove the last inequality (which has important consequences in the theory of convergence of Fourier series) in a simple way by a sort of convexity theorem.

The difficulty of the problem lies in the fact that, although the passage from $f(e^{i\theta})$ to $S_{n(\theta)}(\theta)$ is linear, Marcel Riesz's theorem cannot be applied in the same manner as in §2 above, for the following reason: f is not *any* Fourier series with complex coefficients; it is a power series, and for this reason cannot be approximated by step functions, but only by polynomials of the form $\sum_0^n c_k e^{ik\theta}$.

It is, however, possible to overcome this difficulty, and we have the following general theorem [7]:

Let $M(\alpha, \beta)$ ($\alpha > 0$, $\beta \geqq 0$) denote the maximum of the modulus of the complex bilinear form

$$\sum_{j=0}^{m} \sum_{h=0}^{n} a_{jh} x_j y_h$$

under the conditions $\sum_0^n |y_h|^{1/\beta} \leqq 1$, $\int_0^{2\pi} |f(re^{i\theta})|^{1/\alpha} \leqq 1$ $(0 \leqq r < 1)$ when $f(z) = \sum_0^\infty x_\nu z^\nu$ belongs to $H^{1/\alpha}$ and has $x_0, x_1, \cdots, x_m$ as its first $m+1$ expansion coefficients. Then, if $\alpha = t\alpha_1 + (1-t)\alpha_2$, $\beta = t\beta_1 + (1-t)\beta_2$, one has

$$M(\alpha, \beta) \leqq C(\alpha_1, \alpha_2) M^t(\alpha_1, \beta_1) M^{1-t}(\alpha_2, \beta_2),$$

$C(\alpha_1, \alpha_2)$ being a finite constant depending on α_1 and α_2 only.

This is not exactly a convexity theorem, owing to the introduction of the constant $C(\alpha_1, \alpha_2)$; but its usefulness is the same since it asserts the finiteness of $M(\alpha, \beta)$ when (α, β) lies on the segment $(\alpha_1, \beta_1) - (\alpha_2, \beta_2)$ and when $M(\alpha_1, \beta_1)$ and $M(\alpha_2, \beta_2)$ are both finite.

We shall indicate briefly the ideas underlying the proof. We start from:

$$\left| \sum_{j=0}^{m} \sum_{h=0}^{n} a_{jh} x_j y_h \right| \leqq M(\alpha, \beta) \left[\int_0^{2\pi} | f(e^{i\theta}) |^{1/\alpha} d\theta \right]^{\alpha} \left[\sum_0^n | y_h |^{1/\beta} \right]^{\beta}$$

valid for $\alpha = \alpha_1$, $\beta = \beta_1$ and $\alpha = \alpha_2$, $\beta = \beta_2$, and we want to prove that a similar inequality holds for $\alpha = \alpha_1 t + (1-t)\alpha_2$, $\beta = \beta_1 t + (1-t)\beta_2$. The first natural idea is to prove the theorem when f is a polynomial of degree p, and to replace the integral by a sum of $p+1$ terms, which would allow us to take the elements of this sum (more exactly: the values of the polynomial at the points of subdivision of the interval $(0, 2\pi)$, values which are linear combinations of the x_j) as new variables. The difficulty lies in the fact that the ratio between the integral and the approximating sum lies between two constants only if the exponent $1/\alpha$ stays larger than a constant $q > 1$. Hence we use the idea, which has been used first by Thorin in the proof of a particular result of the same kind [10, pp. 31–35], of writing $f(z) = G^k(z)$. By G we shall denote here a polynomial $\xi_0 + \xi_1 z + \cdots + \xi_p z^p$ and by k the smallest integer such that $k/\alpha_1 \geqq 2$, $k/\alpha_2 \geqq 2$. We can then apply the method just indicated of approximation of the integral by a finite sum, and Thorin's extension of M. Riesz's convexity theorem is essential here, because the x_j become functions $\phi_j(\xi_0, \cdots, \xi_p)$ of the variables $\xi_0, \cdots, \xi_p$ which are homogeneous and of degree k (and thus the function of the ξ's and y's is no longer bilinear).

The inequality once proved for $f = G^k(z)$, G being a polynomial, remains true when f is any function of the class $H^{1/\alpha}$ having no zeros for $|z| < 1$. One has afterwards to remember only that every $f \in H^{1/\alpha}$ is the sum of two zero-free functions f_1 and f_2 of the same class, with $|f_1| \leqq 2|f|$, $|f_2| \leqq 2|f|$.

Finally, if one writes $X_h = \sum_{j=0}^m a_{jh} x_j$ $(h = 0, \cdots, n)$ it is familiar that $M(\alpha, \beta)$ is also the maximum of the ratio

$$\left(\sum_0^n |X_h|^{1/\gamma}\right)^\gamma \Big/ \left[\int_0^{2\pi} |f(e^{i\theta})|^{1/\alpha} d\theta\right]^\alpha$$

where $\gamma = 1 - \beta$. From sums of powers of linear forms we can pass to integrals, and the theorem of Littlewood and Paley is the consequence of our result applied between $\alpha = \gamma = 1/2$ and $\alpha = \gamma = 1$.

The same general result is immediately applicable to the proof of a theorem of Hardy-Littlewood proved directly by Thorin in his thesis [10]. If $f \sim \sum_0^\infty c_\nu e^{i\nu\theta}$, one has:

$$\sum \frac{|c_\nu|}{\nu + 1} \leqq A_1 \int_0^{2\pi} |f|\, d\theta \qquad (f \in H),$$

$$\left(\sum |c_\nu|^2\right)^{1/2} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f|^2 d\theta\right)^{1/2} \qquad (f \in H^2);$$

this means that

$$\left[\sum \frac{1}{(\nu+1)^2}\left[|c_\nu|\,(\nu+1)\right]^{1/\gamma}\right]^\gamma \leqq A(\alpha, \gamma)\left(\int_0^{2\pi} |f|^{1/\alpha} d\theta\right)^\alpha$$

is valid for $\alpha = 1$, $\gamma = 1$ with $A(1, 1) = A_1$; and for $\alpha = 1/2$, $\gamma = 1/2$ with $A(1/2, 1/2) = (2\pi)^{-1/2}$. Hence applying our result we find that, if $1 \leqq p \leqq 2$, there is a constant A_p depending on p only such that

$$\sum \frac{|c_\nu|^p}{(\nu+1)^{2-p}} \leqq A_p \int_0^{2\pi} |f|^p d\theta.$$

Of course, if $1 < p \leqq 2$ (but not for $p = 1$) this is also true when $f \in L^p$ and the c_ν are the complex Fourier coefficients of f.

Finally, let us mention that, using his generalizations of Marcel Riesz's convexity theorem, Thorin, in his thesis [10], has given a new proof (and even an extension to several dimensions) of Hardy and Littlewood's inequality

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{f(x)g(y)}{|x - y|^{2-\alpha-\beta}}\, dxdy\right|$$

$$\leqq K(\alpha, \beta)\left(\int_{-\infty}^{\infty} |f(x)|^{1/\alpha} dx\right)^\alpha \left(\int_{-\infty}^{\infty} |g(y)|^{1/\beta} dy\right)^\beta$$

$$(\alpha + \beta > 1,\ \alpha < 1,\ \beta < 1,\ f \in L^{1/\alpha},\ g \in L^{1/\beta} \text{ in } (-\infty, \infty)).$$

This proof, however, is based not only on a convexity theorem but also on a deep result of Marcinkiewicz in the theory of Fourier series.

REFERENCES

1. G. H. Hardy and J. E. Littlewood, Proc. Cambridge Philos. Soc. vol. 40 (1944) pp. 103–107.

2. J. E. Littlewood and R. E. A. C. Paley, Proc. London Math. Soc. vol. 43 (1937) p. 105.

3. J. Marcinkiewicz and A. Zygmund, *Some theorems on orthogonal systems*, Fund. Math. vol. 28 (1937) pp. 309–335.

4. R. E. A. C. Paley, *A proof of a theorem on bilinear forms*, J. London Math. Soc. vol. 6 (1931).

5. M. Riesz, *Sur les maxima des formes bilineaires et sur les fonctionelles lineaires*, Acta Math. vol. 49 (1927).

6. R. Salem, *Sur une extension du théorème de convexité de M. Marcel Riesz*, Colloquium Mathematicum vol. 1 (1947) pp. 6–8.

7. R. Salem and A. Zygmund, *A convexity theorem*, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 443–447.

8. J. D. Tamarkin and A. Zygmund, *Proof of a theorem of Thorin*, Bull. Amer. Math. Soc. 50 (1944) pp. 279–282.

9. G. O. Thorin, *An extension of a convexity theorem due to M. Riesz*, Kungl. Fysiografiske Sallskapets i Lund Forhandlinger vol. 8 (1939) p. 14.

10. ———, *Convexity theorems*, Meddelanden fran Lunds Universitets Matematiska Seminarium, Thesis, Uppsala, 1948.

11. L. C. Young, *On an inequality of Marcel Riesz*, Ann. of Math. vol. 40 (1939).

12. A. Zygmund, Fund. Math. vol. 30 (1938) p. 190.

13. ———, *Trigonometrical series*, Warsaw, 1935.

14. ———, Prace Matematyczno-Fizyczne vol. 44 (1935).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY