pair of its points $x$, $y$, there exists a homeomorphism between two open subsets of $M$, one containing $x$, the other containing $y$, such that $x$ maps into $y$. Alternatively, the requirement of local connectedness can be replaced by the condition that $M$ contain a simple closed curve. It is incidentally shown that any bounded, plane continuum which is the sum of a collection of disjoint simple closed curves is topologically equivalent to an annulus. (Received March 9, 1949.)

423. O. H. Hamilton: *Transformations topologically equivalent to isometric transformations in Hilbert space.*

It is shown that if $T$ is a pointwise periodic transformation of a point set in an $n$-dimensional Euclidean space, then $T$ is topologically equivalent to a transformation of a point set in a Hilbert space which leaves the distance between two points invariant. (Received March 17, 1949.)

J. W. T. Youngs,
*Associate Secretary*

## THE APRIL MEETING IN STANFORD UNIVERSITY

The four hundred forty-eighth meeting of the American Mathematical Society was held at Stanford University, Palo Alto, California, on Saturday, April 30, 1949. The attendance was approximately 85, including the following 71 members of the Society:

M. I. Aissen, T. M. Apostol, H. A. Arnold, H. M. Bacon, E. M. Beesley, R. L. Belzer, H. F. Bohnenblust, J. L. Botsford, F. A. Butter, Lamberto Cesari, C. L. Clark, F. G. Creese, P. H. Daus, A. C. Davis, E. A. Davis, M. R. Demers, S. P. Diliberto, Roy Dubisch, W. D. Duthie, J. M. G. Fell, G. E. Forsythe, A. L. Foster, R. A. Fuchs, P. R. Garabedian, M. A. Girshick, J. W. Green, Sarah Hallam, G. J. Haltiner, M. A. Heaslet, J. G. Herriot, M. R. Hestenes, R. E. Holdman, R. D. James, Samuel Karlin, J. L. Kelley, R. M. Lackness, Cornelius Lanczos, D. H. Lehmer, Hans Lewy, Eugene Lucacs, Rhoda Manning, W. A. Mersman, A. B. Mewborn, B. C. Meyer, E. D. Miller, F. R. Morris, Ivan Niven, A. M. Ostrowski, Anna Pell-Wheeler, R. S. Phillips, Edmund Pinney, Hugo Ribiero, J. B. Robinson, R. M. Robinson, H. L. Royden, Wladimir Seidel, Max Shiffman, D. C. Spencer, W. M. Stone, Irving Sussman, J. D. Swift, L. H. Swinford, Gabor Szegö, J. L. Ullman, F. A. Valentine, R. L. Vaught, Morgan Ward, D. V. Widder, V. A. Widder, A. R. Williams, František Wolf.

In the morning there was a general session for research papers and for the invited address, *Quadratic forms in the calculus of variations*, by Professor M. R. Hestenes of the University of California, Los Angeles. Professor Max Shiffman presided. In the afternoon there were two sections, pure and applied mathematics, at which Professors D. C. Spencer and Gabor Szegö presided.

After the meetings, those attending were guests at a tea in Stanford Union.

Abstracts of papers presented at the meetings follow. Papers whose abstract numbers are followed by "*t*" were presented by title. Paper 433 was presented by Mr. Shapley.

## Algebra and Theory of Numbers

### 424. T. M. Apostol: *Dedekind sums and their generalizations.*

The Dedekind sums $s(h, k)$ which appear in Rademacher's convergent series for the partition function $p(n)$ satisfy a well known reciprocity law. The author obtains formulas which prove useful for computing these sums. The sums are generalized by considering $s_p(h, k) = \sum_{\mu=1}^{k-1} (\mu/k)\overline{B}_p(h\mu/k)$, where $\overline{B}_p(x)$ is the $p$th Bernoulli function and $s_1(h, k) = s(h, k)$. A reciprocity law for these sums is derived and the method gives a new arithmetic proof of the known law for $s(h, k)$. The generalized sums are related to the functions $G_p(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^{-p}x^{mn}$ much in the same way that $s(h, k)$ is related to the modular form $\eta(\tau)$ where $\log \eta(\tau) = \pi i\tau/12 - G_1(e^{2\pi i\tau})$. Transformation formulas relating $G_p(e^{2\pi i\tau})$ to $G_p(e^{2\pi i\tau'})$ are obtained for odd $p$, $\tau' = (a\tau+b)/(c\tau+d)$ being a modular transformation. The $s_p(h, k)$ appear in these formulas. The $s_p(h, k)$ are also expressed as infinite series related to certain Lambert series and for odd $p \geqq 3$, $s_p(h, k)$ is the Abel-sum of a divergent series. (Received February 7, 1949.)

### 425. Roy Dubisch: *Remarks on isotopy of nilpotent algebras.*

Isotopy (see A. A. Albert, *Non-associative algebras* I, Ann. of Math. vol. 43 (1942) pp. 685–707) is applied to Ghent's classification of nilpotent algebras of order $\leqq 4$ (*A note on nilpotent algebras in four units*, Bull. Amer. Math. Soc. vol. 40 (1934) pp. 331–338) to yield a classification according to isotopy. It is also shown that the number of isotopy classes for any order is always less than the number of equivalence classes. Furthermore, it is shown that, for every order $n$, there exists a nilpotent algebra isotopic to a non-nilpotent algebra. On the other hand, an example is given of a nilpotent algebra, not a zero algebra, which is not isotopic to a non-nilpotent algebra. All of this is for associative algebras. (Received April 30, 1949.)

### 426. D. H. Lehmer: *The evaluation of certain exponential series.*

The series under consideration have $n$th terms of the form $f(n)$ exp $(-an\pi)$ where $a = 1$, $3^{1/2}$, 2 and where the numerical function $f$ is any one of a large class of well known functions including $\sigma_k(n)$, the sum of the $k$th powers of the divisors of $n$, the number $p(n)$ of unrestricted partitions of $n$ and other familiar partition functions, Ramanujan's function $\tau(n)$ and its generalizations (see Bull. Amer. Math. Soc. vol. 54 (1948) p. 829), numbers of representations by sums of squares, and so on. All series are evaluated in closed form in terms of $\pi$, $\Gamma(3/4)$, and $\Gamma(5/6)$ and their powers. Simple examples are $\sum\sigma_3(n) \exp(-2\pi n) = \pi^2\{[\Gamma(3/4)]^8\ 320\}^{-1} - 1/240$, $\sum(-1)^n n\sigma_7(n) \exp(-3^{1/2}\pi n) = 0$, and $\sum p(n) \exp(-\pi(24n-1)/12) = 2^{1/2}\pi^{-1/4}\Gamma(3/4)$. These results are derived from a consideration of the harmonic and equianharmonic cases of elliptic functions. (Received March 18, 1949.)

### 427. Ivan Niven: *A function associated with $\phi(n)$.*

S. S. Pillai (*On a function connected with $\phi(n)$*, Bull. Amer. Math. Soc. vol. 35 (1929) pp. 837–841) has investigated the arithmetic function $R(n)$ which is defined as the least integer $r$ such that $\phi^r(n) = 1$, where $\phi^r(n)$ denotes the $r$th iterate of the Euler

$\phi$-function. This function is investigated in slightly altered form: define $g(n)$ as $R(n)-1$ for $n\geqq 3$, and $g(2)=g(1)=0$. This alteration is to enable one to state the fundamental theorem: $g(ab)=g(a)+g(b)$, unless $a$ and $b$ are both even, in which case $g(ab)=g(a)+g(b)+1$. Various other properties of the function are obtained. (Received March 17, 1949.)

### 428. R. L. Vaught: *The continuous real-valued functions on a completely regular space.* Preliminary report.

The linear algebra $R(X)$ of all continuous real-valued functions on a completely regular topological space $X$ is made into a topological algebra by the introduction of the topology in which convergence means uniform convergence on compact subsets of $X$. It is shown that this topological algebra characterizes $X$, in fact that $X$ is homeomorphic to the set of all continuous homomorphisms of $R(X)$ into the real numbers (with the usual product topology). Certain subalgebras and ideals of $R(X)$ are discussed. It is hoped to use these results to study the possible uniform structures over a completely regular space. (Received April 30, 1949.)

### 429. Morgan Ward: *Divisibility sequences with a rank function.*

Let $(\phi)$: $\phi_0=0$, $\phi_1=1, \cdots, \phi_n, \cdots$ be an integral-valued divisibility sequence. $(\phi)$ is said to admit a rank function if for every modulus $m$ there exists an integer $r=r(m)$ such that $\phi_n\equiv 0 \pmod{m}$ if and only if $n\equiv 0 \pmod{r}$. Two new necessary and sufficient conditions for $(\phi)$ to admit a rank function are obtained. The simpler of these is as follows: If $p$, $q$ are any two primes, then for every $n$, the greatest common divisor of $\phi_{pn}$ and $\phi_{qn}$ is $\phi_n$. (Received March 18, 1949.)

## ANALYSIS

### 430. E. M. Beesley. *Concerning total differentiability of functions of class P.*

Since 1900, some eight or more writers have formulated conditions under which a function of two real variables shall be said to be of bounded variation. Adams and Clarkson (Trans. Amer. Math. Soc. vol. 35 (1933) pp. 824–854; vol. 36 (1934) pp. 711–730; vol. 46 (1939) p. 468) have examined and compared these definitions and have catalogued many of the properties of the functions which satisfy the various definitions. In the terminology of Adams and Clarkson, $P$ is the class of functions which are of bounded variation in the sense of Pierpont (see, for example, Adams and Clarkson, loc. cit., or H. Hahn, *Theorie der Reelen Funktionen*, Berlin, 1921, p. 539). One of the questions left open by these writers is the following: Do there exist functions of class $P$ which are not totally differentiable almost everywhere? It is shown that there are continuous functions of class $P$ which are nowhere totally differentiable. A specific example is exhibited and several questions concerning the category of the set of these functions in various function sets are considered. (Received March 18, 1949.)

### 431. Lamberto Cesari: *On Geocze's problem.*

Given a Fréchet surface $S$: $x=x(u, v)$, $y=y(u, v)$, $z=z(u, v)$, $(u, v)\in A$ ($A$ a Jordan region), the Lebesgue area $L(S)$ of $S$ is the lim inf of the elementary areas $a(\Sigma)$ of all polyhedral surfaces $\Sigma$ for which $\|S, \Sigma\|\rightarrow 0$. $L^*(S)$ is the corresponding lim inf when $\Sigma$ is restricted to those $\Sigma$ inscribed in $S$. $L^*(S)\geqq L(S)$; Geocze's problem is to prove

$L^*(S) = L(S)$. If the proof is developed only in terms of the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ of the given representation of the surface $S$, this is the so-called strong form of the Geocze problem (Radó, Huskey); if it is permitted to use in the proof other and convenient representations of the same surface $S$, this is the weak form of the problem (Mambriani). The author has proved, in the weak form and utilizing his previous research on representation of surfaces, that for each Fréchet surface $S$, $L^*(S) = L(S)$. (Received April 30, 1949.)

### 432. R. D. James: *A generalized integral.* II.

The definition and some of the properties of what may be called a Perron second integral ($P^2$-integral) were given in a previous paper (Transactions of the Royal Society of Canada (3) vol. 40 (1946) pp. 25–35). In the present paper the definition is changed slightly and further properties are deduced. It is shown, for example, that the $P^2$-integral provides a complete solution to a problem of Denjoy which he discussed in five notes in the C. R. Acad. Sci. Paris in 1921 (vol. 172, p. 653, p. 833, p. 903, p. 1218; vol. 193, p. 127). It is also proved that if a trigonometric series converges in $(0, 2\pi)$ to a function $f(x)$, then $f(x)$ is $P^2$-integrable, and that, with a suitable modification of the definition of the Fourier coefficients for the $P^2$-integral, the trigonometric series is the Fourier series of $f(x)$. (Received March 17, 1949.)

### 433. Samuel Karlin and L. S. Shapley: *Moment spaces and polynomial games.*

The reduced moment space $D^n$ is the set of points in $E^n$ whose coordinates are the first $n$ moments of some distribution function $\phi(t)$ over the interval $[0, 1]$. $D^n$ is the convex closure of the skew curve $x_i = t^i$, $0 \leqq t \leqq 1$, $i = 1, \cdots, n$. Boundary points of $D^n$ may be classified according to facial dimension: let $a(x)$ denote the dimension of the intersection $L(x)$ of all planes of support at $x$. The boundary has two faces of each dimension $a = 0, \cdots, n-1$. Each $x$ in the boundary is uniquely representable by extreme points and corresponds to a unique distribution $\phi_x$. The number of points in the representation of $x$ and in the spectrum of $\phi_x$ is the same, and is denoted by $b(x)$, or by $\bar{b}(x)$ if the points 0 and 1 are counted with half weight when they appear. Then $a(x) = 2\bar{b}(x) - 1$ and $b(x) = c(x) + 1$, where $c(x)$ is the dimension of $L(x) \cap D^n$. The moment-space game with strategies $x \in D^m$, $y \in D^n$, and pay-off $\sum a_{ij} x_i y_j$ is equivalent to the polynomial game with strategies $t$, $u \in [0, 1]$ and pay-off $\sum a_{ij} t^i u^j$; the mixed strategies of the latter corresponding to the pure strategies of the former. The foregoing considerations reveal many facts about the solutions (best mixed strategies) of polynomial games. (Received April 30, 1949.)

### 434. Cornelius Lanczos: *An iteration method for solving the eigenvalue problem of linear differential operators.*

Let $Df(x)$ be a given linear differential operator of a given finite range, with linear homogeneous boundary conditions which establish an eigenvalue problem. Expand $y = f(x)$ into a finite series of linearly independent functions, such as the powers of $x$. Determine the coefficients of this expansion by the condition that not only $y$, but also $Dy$, $D^2y$, $\cdots$, $D^my$ shall satisfy the given boundary conditions. Then an approximate representation of the first $k$ eigenfunctions of the operator $Dy$ is obtainable as a given linear superposition of $y$, $Dy$, $\cdots$, $D^{k-1}y$; $(k < 2m)$. With increasing $m$ ($k$ fixed) the error converges to zero. Compared with the Rayleigh-Ritz method the following advantages may be listed: (1) The self-adjointness of $Dy$ is not required. (2) No integration is involved, only the solution of linear equations. (3) Instead of successive reductions, a whole set of eigenfunctions and eigenvalues can be obtained simultaneously. The convergence is satisfactory if proper care is taken in the choice of the class of functions into which $y$ is expanded. (Received March 17, 1949.)

### 435t. Walter Leighton: *Principal quadratic functionals.*

A principal quadratic functional is defined to be a functional of the form $J=\int_0^b[r(x)y'^2-p(x)y^2]dx$, where $b>0$, $r(x)$ and $p(x)$ are continuous, and $r(x)>0$ on $0<x\leqq b$. The minimizing of $J$ among various classes of admissible curves is studied, and necessary and sufficient conditions are obtained. This paper, which will appear in the Trans. Amer. Math. Soc., may be regarded as a continuation of investigations begun by Morse and Leighton (*Singular quadratic functionals*, Trans. Amer. Math. Soc. vol. 40 (1936) pp. 252–286). (Received March 7, 1949.)

### 436. W. A. Mersman: *A new form of solution of Hermite's equation.*

In a certain compressible laminar boundary-layer investigation it is found necessary to solve the differential equation $d^2y/dz^2+2zdy/dz-4ny=0$, $n$ being any non-negative integer, subject to the boundary condition that $y(z)$ vanish at infinity. Although this equation is readily identifiable with the Hermite equation of order $2n$ with imaginary independent variable, the standard form of the classical solution is of no practical use in satisfying the condition at infinity. In the present paper, a general solution is obtained as the sum of three terms: (1) a polynomial, (2) a polynomial multiplied by the error function, and (3) a polynomial multiplied by exp $(-z^2)$. This form of solution is amenable to numerical computations for all values of $z$, whether large or small, and the condition at infinity is easily satisfied. (Received March 14, 1949.)

## Applied Mathematics

### 437t. H. W. Becker: *A new proof and application of Kirchhoff's rules.*

Philip Franklin (Journal of Mathematics and Physics (1925) pp. 97–102) gave a determinantal proof of Kirchhoff's rules for assembling a network function, by excluding terms whose complementaries contain meshes. A proof is here given, by ultimate fracto decomposition of the circuit into nothing but opens and shorts, using all branches as keys. Duality yields another rule, excluding terms containing modes, proper or *improper* (pairs or sets of adjacent modes). The rules are applied to the $Z$ solution of all 16 fundamental bridge networks of from 6 to 11 unit branches in R. M. Foster's enumeration (Transactions of the American Institute of Electrical Engineers (1932) pp. 309–317) and to all 54 fundamental passive bridge circuits therein imbedded. If $Z=(za+b)/(zc+d)$ is the driving point impedence of a circuit, and $Z_z=(za+b)/f$ is the transfer impedence from the source to branch $z$, then the transfer factor $f=(ad-bc)^{1/2}$ entails $(f)=a\frown d$, where $(f)$ is the set of terms of $f$ without regard to sign. That is, $a=(f)+\alpha$, $d=(f)+\delta$, so the source $\leftrightarrow z$ interchange factor $h=a-d$ is irreducibly $h=\alpha-\delta$. (Received March 7, 1949.)

### 438t. H. W. Becker: *Circuitoidal functions.*

These are quotients $Z=D/N$, where $D$ is a partial sum of products of $n$ variables $\mu$ at a time, and $N$ is a p.s.p. of the $n$ or less variables $\mu-1$ at a time. The number of such possible quotient forms $CF(n)=3[(n-1)!]^2$, $n=2, 3, 4$. Those C.F. which are not driving point or transfer impedences of any physical circuit (and for $n \geqq 4$, such are in the vast majority) have the status of imaginary circuits. The simplest is Cohn's i.c., $Z_C = z$, $(w+v) \overset{+}{+} (w \| v) = (zw+zv+wv)/(z+w+v)$. The i.c. may be a fracto imaginary (the fracto sum of physical circuits having consistent multiplicity but not topology) or a semi-imaginary (the quotient of physical semicircuits having c.m.n.t.) or both f.i. and s.i. or neither of them. $Z_C$ is both f.i. and s.i., and so sometimes is $^{-}Z_n$ the dual of a nonplanar circuit. If $S$ is a set and $\Omega$ an operation such that $M$ and $m$ being any two members of $S$, $m' = M\Omega m$ is *not* a member of $S$, then $S$ is a exogroup (exogamous group) with respect to $\Omega$. If $S$ is the infinite class of proper series-parallel circuits (having no opened or shorted branches), then $S$ is an exogroup with respect to fracto addition, $\overset{+}{+}$. (Received March 14, 1949.)

439. A. M. Ostrowski: *A problem of abstract algebra connected with Horner's rule.*

In applying Horner's rule for computing the values of polynomials to polynomials in several variables, $N-1$ multiplications and the same number of additions are to be carried out, where $N$ is the number of terms in the corresponding "complete" polynomial. The question whether for polynomials in one variable this number is the best is discussed and for the first four degrees solved. (Received April 30, 1949.)

440. R. S. Phillips: *The electromagnetic field produced by a helix.*

Starting with the retarded vector potential produced by a sinusoidal current traveling along an infinitely-thin perfectly-conducting helix, the phase velocity of this current is determined so that the electric field along the helix is "essentially" zero. The phase velocity along the helix so obtained is equal to the free space velocity of propagation. A Bessel function expansion is found for the electromagnetic field from integral evaluations of the type $\int_{-\infty}^{\infty} e^{i\beta\zeta}(e^{ikR}/R)d\zeta = 4\pi \sum_{n=-\infty}^{\infty} e^{in\phi}(I_n(\Gamma_n a)K_n(\Gamma_n \rho)$ where $R = (\zeta^2 + a^2 + \rho^2 - 2a\rho \cos(\phi - \zeta/\alpha))^{1/2}$, $\Gamma_n = ((\beta - n/\alpha)^2 - K^2)^{1/2}$ and $\rho > a$. In order to determine the phase velocity, it is shown that $\lim_{\rho \to a^+} \Omega(\beta, \rho) = \infty$ and $\lim_{\rho \to a^+} \Omega(\beta_1, \rho)/\Omega(\beta_2, \rho) = 1$ for any real $\beta_1$ and $\beta_2$, where $\Omega(\beta, \rho) = \sum_{n=-\infty}^{\infty} I_n(\Gamma_n a)K_n(\Gamma_n \rho)$. (Received February 16, 1949.)

441. Edmund Pinney: *Elastic vibrations due to a point source within a semi-infinite solid.*

$P$ or $S$ waves originating at a point within a semi-infinite solid are investigated. Surface accelerations are calculated. (Received March 21, 1949.)

## GEOMETRY

442*t*. D. B. Dekker: *Hypergeodesic curvature and torsion.*

This paper deals with the differential equation of an arbitrary family of hypergeodesics on a surface and definitions of hypergeodesic curvature and hypergeodesic torsion relative to the family in such a manner that the definitions reduce to those for union curvature and union torsion when the family is a family of union curves. The differential equation is first expressed in a tensor form as the vanishing of a certain scalar for any curve of the family. Then this single differential equation is transformed into two equations stating that a certain vector shall have zero components for any curve of the family. Hypergeodesic curvature and the hypergeodesic curvature vector are defined as the above mentioned scalar and vector respectively. A study of the cusp cone provides a geometric interpretation of hypergeodesic curvature of an arbitrary curve on the surface. An expression for the torsion of a hypergeodesic yields a definition of hypergeodesic torsion of a curve and leads to the theorem that a hypergeodesic is a plane curve if and only if the elements of contact of the osculating planes with the corresponding cusp cones along the curve form a developable. (Received March 14, 1949.)

### 443. J. W. Green: *Sets subtending a constant angle on a circle.*

Let $C$ be a closed circular area in the plane and $C'$ its circumference. Let $K$ be a convex set in $C$ which subtends at each point of $C'$ the constant angle $\alpha$, $0<\alpha<\pi$. The paper considers the possibility of the existence of non-circular sets $K$ with this property. It is found that if $\alpha$ is an irrational multiple of $\pi$ or if $\alpha=(m/n)\pi$, $m$ and $n$ relatively prime and odd, there is only the obvious concentric circle to $C$ as solution; in all other cases there is a large class of solutions. In the case where non-circular solutions are possible, a number of extremal problems are solved, involving the length, width, diameter, and area of these sets. For example, among all convex sets in a circle subtending $\pi/2$ on its circumference (and there are many, including a one-parameter family of ellipses) the circle has the greatest length. The minimum length is not attained but the limiting case is a diameter of $C$. (Received March 17, 1949.)

### 444. F. A. Valentine: *Some properties of C-convex sets.*

A set is said to be $C$-convex if each of its components is convex. A $C$-convex set having $n$ components is called a $C_n$-convex set. A set $S$ is called an $L_n$ set if each pair of points in $S$ can be joined by a polygonal line in $S$ having at most $n$ segments. The author shows that in the plane an open, bounded $C_n$-convex set $(n>1)$ is an $L_{n+1}$ set. In order to prove this, use is made of the notion of maximal family. A family of $n$ disjoint open convex sets $M_n$ is said to be maximal if no member of the family is a proper subset of an open convex set which is disjoint with the remaining members of the family. The boundary of an open maximal family $M_n$ is the sum of linear edges, and it may or may not be connected. In the plane, each component of the complement of the closure of an open maximal family $M_n$ is a convex set bounded by a closed polygon. Further properties of such sets are obtained and applied to the study of $C_n$-convex sets. (Received March 17, 1949.)

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