tive) ring, H a subring of K and that (α) H has a 1-element; (β) the equation $xh=h_1$ with h, $h_1 \in H$, $x \in K$, $h \neq 0$ implies that $x \in H$, (γ) for every $a \in K$, $b \in H$, there exists an element b_1 in H with $ba=ab_1$. If $H \neq K$, it follows that every element of H commutes with every element of K.

UNIVERSITY OF TORONTO

A THEOREM ON INTEGRAL SYMMETRIC MATRICES[1]

B. W. JONES

Though the following theorem yields important results in the theory of quadratic forms, its statement and proof are independent of such theory and seem to possess significance in their own right.

THEOREM. *Let A and B be symmetric integral nonsingular matrices with respective dimensions n and m ($n>m$) and S an n by m matrix of rank m with rational elements such that s is the* l.c.m. *of the denominators and $S^TAS=B$. Then there is an n by n matrix T with rational elements the prime factors of whose denominators all divide s, whose determinant is* 1 *and which takes A into an integral matrix A_0 which represents B integrally, that is, $U^TA_0U=B$ for some integral matrix U.*

To prove this we first, for brevity's sake, define an *s-matrix* or *s-transformation* to be one with rational elements the prime factors of whose denominators all divide s. Then write $R=sS$, and, by elementary divisor theory, determine unimodular matrices P and Q such that

$$PRQ = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = s\begin{bmatrix} S_1 \\ 0 \end{bmatrix} = sS'$$

where R_1 is the diagonal matrix $r_1 \dotplus r_2 \dotplus \cdots \dotplus r_m$, $\dotplus$ denotes direct sum, r_i divides r_{i+1} for $i=1, 2, \cdots, m-1$ and S' and S_1 are defined by the equations. Write $r_i/s=u_i/s_i$ where the latter fraction is in lowest terms and $s_i>0$. Then s_i is divisible by s_{i+1} and hence s_i is prime to u_j for $j \leqq i$.

Presented to the Society, September 10, 1948; received by the editors June 7, 1948.

[1] This paper was written while on sabbatical leave from Cornell University with the aid of a grant from the Research Corporation.

If we write $A'=P^{IT}AP^I$ and $B'=Q^TBQ$, the equality $S^TAS=B$ becomes $S'^TA'S'=B'$ which, with $A'=(a_{ij})$ and $B'=(b_{ij})$, implies

$$a_{ij}r_ir_j/s^2 = a_{ij}u_iu_j/s_is_j = b_{ij}, \qquad i, j = 1, 2, \cdots, m. \tag{1}$$

Suppose $s_i=1=|u_i|$ for $i=1, 2, \cdots, h-1$ but not for $i=h$. Then (1) and B' integral implies that s_h divides a_{ih} for $1\leqq i\leqq h$ and s_h^2 divides a_{hh}. Moreover u_h divides b_{ih} for $1\leqq i\leqq m$ and u_h^2 divides b_{hh}. Write A' in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

where A_{11} is an h by h matrix. Let D_h be the matrix obtained from the h-rowed identity matrix by replacing its last diagonal element by s_h and U_h the matrix obtained from the m-rowed identity matrix by replacing its hth diagonal element by u_h. Write $F_h=D_h^I\dot{+}K_h$ where K_h is an integral $(n-h)$-rowed square matrix of determinant s_h later to be determined. We then have

$$F_h^TA'F_h = \begin{bmatrix} D_h^IA_{11}D_h^I & D_h^IA_{12}K_h \\ K_h^TA_{12}^TD_h^I & K_h^TA_{22}K_h \end{bmatrix} = A_h.$$

Then S_h, defined by the equation $S_h=F_h^IS'U_h^I$, is the direct sum of the integral matrix $u_1\dot{+}u_2\dot{+}\cdots\dot{+}u_{h-1}\dot{+}1$, whose elements have absolute value 1 and an $n-h$ by $m-h$ s-matrix. Furthermore $S'^TA'S'=B'$ becomes $S_h^TA_hS_h=B_h$ where the equality $U_h^IB'U_h^I=B_h$ defines B_h. Now B_h is integral since, as shown above, u_h divides b_{ih} for $1\leqq i\leqq m$ and u_h^2 divides b_{hh}. Furthermore B_h represents B' integrally. Similarly $D_h^IA_{11}D_h^I$ is integral.

Next we determine K_h consistent with the above conditions so that A_h is integral, that is, so that $D_h^IA_{12}K_h$ is integral. In fact, in view of the definition of D_h, we need only make αK_h divisible by s_h where α is the hth row of A_{12}. This is easily done by finding a unimodular matrix W so that $\alpha W\equiv(w, 0, \cdots, 0) \pmod{s_h}$ and choosing W^IK_h to be the diagonal matrix $s_h\dot{+}1\dot{+}\cdots\dot{+}1$ of determinant s_h.

If now we diagonalize the last $n-h$ rows of S_h replacing A_h and B_h by equivalent matrices (that is, matrices obtained from them by unimodular transformations) we may continue along the above lines to derive inductively a sequence of integral matrices A_h obtained from A by s-transformations of determinant 1 and taken by transformations S_h into integral matrices B_h which represent B' integrally, each S_h being the direct sum of an h-rowed integral matrix and an s-matrix.

Then S_m is integral and taking $A_m = A_0$, $B_m = B_0$ we see that A_0 is obtained from A by an s-transformation of determinant 1, represents B_0 integrally and hence B integrally.

Since two quadratic forms with integral coefficients may be defined to be in the same genus if one may be taken into the other by a transformation of determinant ± 1 with rational elements whose denominators are prime to twice the determinant of the form, we have the following

COROLLARY 1. *If f and g are two quadratic forms with integral coefficients, having nonzero determinants and n and m variables respectively, $n > m$, and if there is an n by m matrix with rational elements whose denominators are prime to $2|f|$ and taking f into g, then there is a form f_0 in the same genus as f whose matrix has integral elements and which represents g integrally.*

This corollary has been proved using the theory of quadratic forms by C. L. Siegel,[2] whose proof contained some ideas in common with the above proof, by Hel Braun[3] and probably follows from the much earlier work of Minkowski. However in all these sources the condition that the denominators be prime to $2|f|$ is essential. It is then of interest to note that this restriction is not necessary for the theorem of this note.

In fact, a direct consequence of our theorem (as noted by the referee) is

COROLLARY 2. *If a classic quadratic form f (that is, a form whose matrix has integer elements) represents an integer or classic form g rationally, then some classic form f' of the same determinant as f represents g integrally.*

Thus for example the conditions for integral representation by the form $x_1^2 + x_2^2 + \cdots + x_n^2$, $1 \leqq n \leqq 7$, are the same conditions as those for rational representation since there is only one class of forms of determinant 1 in these cases.

CORNELL UNIVERSITY

---

[2] C. L. Siegel, *Über die analytische Theorie der quadratischen Formen*, Ann. of Math. (2) vol. 36 (1935) pp. 527–606, Lemma 24.

[3] Hel Braun, *Geschlechter quadratischer Formen*, J. Reine Angew. Math. vol. 182 (1940) pp. 32–49.