INTEGRAL EXTENSIONS OF A RING

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Introduction. Let R be a commutative ring with a unit element and let $a, b \subset R$.

DEFINITIONS. (1) a and b are said to be coprime in R if $\tau \subset R$, τ/a, τ/b implies $\tau/1$.

(2) A ring R' is called an integral extension of R if $R' \supset R$ and $a = br$, a, $b \subset R$, $r \subset R'$ implies there exists an element $\bar{r} \subset R$ such that $a = b\bar{r}$.

(3) a and b are said to be absolutely coprime if they are coprime in every extension R' of R. In this paper it is shown that to every set of ideals of a commutative ring there exists an extension of the ring such that every ideal of the set is the intersection of the ring and a principal ideal of its extension. This is the main result and is given in Theorem 2. In a particular case of Theorem 2 it is shown in Theorem 1 that $a, b \subset R$ are absolutely coprime if and only if there exist elements $x, y \subset R$ such that $ax + by = 1$.

Similar results for algebraic integers are well known [**1**].[1] In the special case where the domains considered are completely integrally closed and the ideals have finite bases, a different extension fulfilling the conditions of Theorem 2 was obtained by Kronecker [**2**].

The extension of R. An extension of R, in the sense of this paper, may be obtained in the following manner. We first form the ring $R(u)$ by adjoining to R the elements u^n, $n = \pm 1, \pm 2, \cdots$, transcendental over R and such that $u^n a = a u^n$, $a \subset R$. Let $\mathfrak{a}$ be the ideal generated by the set A of elements $a, b, \cdots$ of R. Then the subring R' of $R(u)$ consisting of all "polynomials"

$$a_{-m}u^{-m} + a_{-m+1}u^{-m+1} + \cdots + a_{-1}u^{-1} + a_0 + a_1 u + \cdots + a_n u^n$$

with $a_i \subset R$ and $a_{-r} \subset \mathfrak{a}^r$, $r > 0$, is an integral extension of R for $R' \supset R$. Moreover if $c = dh$, c, $d \subset R$, $h \subset R'$, then

$$h = e_{-m}u^{-m} + \cdots + e_{-1}u^{-1} + e_0 + e_1 u + \cdots + e_n u^n$$

$e_i \subset R$, $e_{-r} \subset \mathfrak{a}^r$. Multiplying by d we have

$$c = de_{-m}u^{-m} + \cdots + de_{-1}u^{-1} + de_0 + de_1 u + \cdots + de_n u^n.$$

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[1] Numbers in brackets refer to the references cited at the end of the paper.

Since $c \subset R$, all terms involving u must vanish. Hence $c = de_0$ and R' is an integral extension of R.

Results.

THEOREM 1. *a, $b \subset R$ are absolutely coprime if and only if there exist elements x, $y \subset R$ such that $ax+by=1$.*

Clearly the condition is sufficient. Use the above extension R' of R where A consists of a and b. In $R'=R(au^{-1}, bu^{-1}, u)$, u is a common divisor of a and b. If a and b are absolutely coprime, u must be a unit divisor. Hence

$$u^{-1} = \alpha_{-m}u^{-m} + \cdots + \alpha_{-1}u^{-1} + \alpha_0 + \alpha_1 u + \cdots + \alpha_n u^n$$

$\alpha_i \subset R$, $\alpha_{-r} \subset \mathfrak{a}^r$. Since u is trancendental over R all terms in the sum must vanish except $\alpha_{-1}u^{-1}$. But $\alpha_{-1} \subset \mathfrak{a}$ and hence is of the form $ax+by$. Multiplying by u we have

$$1 = ax + by.$$

REMARK. If, $\bar{x}$, $\bar{y}$ are elements of R such that

$$a\bar{x} + b\bar{y} = 1$$

then the pair x, $y \subset R$ is also a solution if and only if $x = \bar{x} + b\mu$, $y = \bar{y} - a\nu$ where $ab\mu = ab\nu$.

Substitution shows the condition to be sufficient. Moreover if $ax+by=1$, $a\bar{x}+b\bar{y}=1$ then $a(x-\bar{x})=b(\bar{y}-y)$ and so $a\bar{x}(x-\bar{x}) = b(\bar{y}-y)\bar{x}$. Adding $b\bar{y}(x-\bar{x})$ to each side of the last equation we have

$$x - \bar{x} = b(\bar{y} - y)\bar{x} + b\bar{y}(x - \bar{x}) = b\mu.$$

Similarly $y - \bar{y} = -a\nu$.

But $a(x-\bar{x}) = b(\bar{y}-y)$. Hence

$$ab\mu = ab\nu.$$

LEMMA. *If $\mathfrak{a}$ is an ideal in R, then there exists an extension R' of R such that $\mathfrak{a}$ is the intersection of R and a principal ideal of R'.*

Consider the extension R' of R with $\mathfrak{a} = A$. In R' the ideal (u) is principal and contains $\mathfrak{a}$ and no other elements of R, for every element of $\mathfrak{a}$ is obtained from the set of products $au^{-1}xu$. Also for $\lambda \subset R'$ suppose $\lambda u = c$, $c \subset R$. Then $\lambda = cu^{-1}$. Hence $c \subset \mathfrak{a}$ and $\mathfrak{a} = R \wedge (u)$.

THEOREM 2. *To every set of ideals of R there exists an extension R' of R such that every ideal of the set is the intersection of R and a principal ideal of R'.*

To each ideal $\mathfrak{a}$ in the set let there correspond an element $u_{\mathfrak{a}}$ transcendental over R. Form the ring $R(u)$ by adjoining $u_{\mathfrak{a}}^n$ to R where $u_{\mathfrak{a}}^n a = a u_{\mathfrak{a}}^n$, $a \subset R$, $n = \pm 1, \pm 2, \cdots$. The subring R' of $R(u)$ consisting of the "polynomials"

$$\sum a_{r_1 r_2 \ldots} u_1^{r_1} u_2^{r_2} \cdots, \qquad a_{r_1 r_2 \ldots} \subset R,$$

with the condition that $a_{r_1 r_2 \cdots r_i}$ belong to $\mathfrak{a}_i^{-r_i}$ if r_i is negative, is an integral extension of R. This follows by the method demonstrated above. For if $l = mn$, l, $m \subset R$, $n \subset R'$ then

$$n = \sum \beta_{r_1 r_2 \ldots} u_1^{r_1} u_2^{r_2} \cdots$$

where $\beta_{r_1 r_2 \cdots}$ are certain $a_{r_1 r_2 \cdots}$. Hence

$$l = mn = \sum m \beta_{r_1 r_2 \ldots} u_1^{r_1} u_2^{r_2} \cdots.$$

But since the indeterminates u_i are transcendental over R and $l \subset R$, all terms in the sum, except the constant term $m\beta_0$, must vanish. Hence $b = m\beta_0$.

We proceed as in the lemma. Consider the principal ideal $(\mathfrak{a})$. All of the elements $a^{(i)}$ of $\mathfrak{a}$ may be obtained as products $a^{(i)} u_{\mathfrak{a}}^{-1} \cdot u_{\mathfrak{a}}$. Moreover only the elements $a^{(i)} \subset R$ may be so obtained for suppose $\nu \subset R'$ and $\nu u_i = d \subset R$. Then $\nu = d u_i^{-1}$. Hence $\mathfrak{a} = R \wedge (u_i)$.

Remark. In the case of non-commutative rings a result analogous to Theorem 2 holds for two-sided ideals.

References

1. H. Prufer, *Neue Begrundung der algebraischen Zahlen Theorie*, Math. Ann. vol. 94 (1925) pp. 198–244.

2. W. Krull, *Idealtheorie*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 1935, pp. 124–127.

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