REMARKS ON CYCLIC ADDITIVITY

J. W. T. YOUNGS[1]

1. Introduction. For the purposes of this discussion suppose that X and Y are topological spaces while G is a commutative, topological semi-group (with zero element) which, as a space, is Hausdorff. In other words, each pair (g_1, g_2) of elements in G uniquely determines an element (g_1+g_2) in G; the operation $+$ is associative and commutative; there is a unique element 0 such that $g \in G$ implies $g+0=g$; finally, G is a Hausdorf space and the operation $+$ provides a mapping (=continuous transformation) from the product space $G \times G$ into G. Obviously, topological groups, and the space of non-negative real numbers compactified by the addition of ∞, with the operation $+$ meaning addition, and the convention that $a+\infty=\infty+a=\infty$, provide examples of such semi-groups.

It will be said that lm is a *Peanian factorization* of a mapping $f: X \to Y$ if and only if there are mappings $m: X \to \mathfrak{X}$ and $l: \mathfrak{X} \to Y$ such that $\mathfrak{X}$ is a Peano space and the composition lm is f. The space $\mathfrak{X}$ is called the *middle space* of the Peanian factorization lm of f.

Let $\boldsymbol{F}$ be the class of mappings $f: X \to Y$ each of which has at least one Peanian factorization, and suppose that γ is a transformation from $\boldsymbol{F}$ into G.

For each Peano space $\mathfrak{X}$ let $\mathcal{E}(\mathfrak{X})$ be the class of true cyclic elements of $\mathfrak{X}$. (See Whyburn [6] for the Peano space theory involved in this paper.)[2] If $\mathfrak{C} \in \mathcal{E}(\mathfrak{X})$ there is a unique monotone retraction $r_{\mathfrak{C}}: \mathfrak{X} \rightrightarrows \mathfrak{C}$. (The double arrow indicates that $r_{\mathfrak{C}}(\mathfrak{X}) = \mathfrak{C}$.) If $f \in \boldsymbol{F}$ and lm is a Peanian factorization of f with middle space $\mathfrak{X}$, while $\mathfrak{C} \in \mathcal{E}(\mathfrak{X})$, then define $f_{\mathfrak{C}} = l r_{\mathfrak{C}} m$.

It is the object of this paper to investigate the statement

$$\gamma(f) = \sum \gamma(f_{\mathfrak{C}}), \qquad \mathfrak{C} \in \mathcal{E}(\mathfrak{X}) \tag{1}$$

where the equality means that for each neighborhood U of $\gamma(f)$ there is a finite subclass $\mathcal{J}(U)$ of $\mathcal{E}(\mathfrak{X})$ such that if $\mathcal{J}$ is any finite subclass of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U)$ then $U \ni \sum \gamma(f_{\mathfrak{C}})$, $\mathfrak{C} \in \mathcal{J}$, it being understood that addition over an empty class yields 0.

In the event that (1) holds for each $f \in \boldsymbol{F}$ and for each Peanian factorization of f, then γ is said to be *cyclicly additive*.

Cyclic additivity theorems of a weaker type have been considered

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[1] Fellow of the John Simon Guggenheim Memorial Foundation.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

by Helsel [3], Radó [5] and others in connection with Lebesgue area.

2. The theorems. *The first object is to investigate consequences of the assumption that γ is cyclicly additive.*

I. If $f \in F$ and has a Peanian factorization in which the middle space is a dendrite, then $\gamma(f) = 0$.

This is an obvious consequence of the fact that a dendrite has no true cyclic elements.

I_a. If $f \in F$ and has a Peanian factorization in which the middle space is an arc, then $\gamma(f) = 0$.

(It is understood that an arc may consist of a single point.)

Before considering more substantial necessary conditions, suppose that $\mathfrak{X}$ is a Peano space and define $\mathfrak{P} \in \mathcal{P}(\mathfrak{X})$ if and only if $\mathfrak{P}$ is a Peano subspace of $\mathfrak{X}$ such that $\mathcal{E}(\mathfrak{P}) \subset \mathcal{E}(\mathfrak{X})$. Let $\mathfrak{Q} \in \mathcal{Q}(\mathfrak{X})$ if and only if $\mathfrak{Q}$ is a Peano subspace of $\mathfrak{X}$ having a finite cyclic chain approximation $\mathfrak{C}_1, \cdots, \mathfrak{C}_q$ such that either $\mathfrak{C}_k$ is an arc, or $\mathfrak{C}_k \in \mathcal{E}(\mathfrak{X})$, $k = 1, \cdots, q$. (See Whyburn [6, p. 73].)

It is easy to see that $\mathcal{Q}(\mathfrak{X}) \subset \mathcal{P}(\mathfrak{X}) \supset \mathcal{A}(\mathfrak{X})$, where $\mathcal{A}(\mathfrak{X})$ is the class of A-sets in $\mathfrak{X}$.

Given $\mathfrak{P} \in \mathcal{P}(\mathfrak{X})$, let $\mathfrak{A} = \mathfrak{P} \cup \bigcup \mathfrak{C}$, where the union is taken over those elements $\mathfrak{C} \in \mathcal{E}(\mathfrak{X})$ having the property that $\mathfrak{P} \cap \mathfrak{C}$ consists of at least two points. It follows that $\mathfrak{A} \in \mathcal{A}(\mathfrak{X})$, that $x \in \mathfrak{A} - \mathfrak{P}$ implies there is a unique $\mathfrak{C} \in \mathcal{E}(\mathfrak{A})$ such that $\mathfrak{x} \in \mathfrak{C}$, and if $\mathcal{E}(\mathfrak{P}) \not\ni \mathfrak{C} \in \mathcal{E}(\mathfrak{A})$ then $\mathfrak{C} \cap \mathfrak{P}$ is a dendrite. Since a dendrite is an absolute retract if $\mathcal{E}(\mathfrak{P}) \not\ni \mathfrak{C} \in \mathcal{E}(\mathfrak{A})$, then there is a retraction $\rho_{\mathfrak{C}}: \mathfrak{C} \rightrightarrows (\mathfrak{C} \cap \mathfrak{P})$. (See Borsuk [1].) Define

$$\theta_{\mathfrak{P}}(\mathfrak{x}) = \begin{cases} \mathfrak{x} & \text{if } \mathfrak{x} \in \mathfrak{P}, \\ \rho_{\mathfrak{C}}(\mathfrak{x}) & \text{if } \mathfrak{x} \in \mathfrak{C} \text{ and } \mathcal{E}(\mathfrak{P}) \not\ni \mathfrak{C} \in \mathcal{E}(\mathfrak{A}). \end{cases}$$

It follows that $\theta_{\mathfrak{P}}$ is a retraction from $\mathfrak{A}$ onto $\mathfrak{P}$, but $\theta_{\mathfrak{P}}$ is not uniquely defined in terms of $\mathfrak{A}$ and $\mathfrak{P}$. If $r_{\mathfrak{A}}: \mathfrak{X} \rightrightarrows \mathfrak{A}$ is the unique monotone retraction, and $r_{\mathfrak{P}} = \theta_{\mathfrak{P}} r_{\mathfrak{A}}$, then $r_{\mathfrak{P}}: \mathfrak{X} \rightrightarrows \mathfrak{P}$ is a retraction.

The retraction $r_{\mathfrak{P}}$ depends upon the retraction $\theta_{\mathfrak{P}}$ which is not unique, however, if $\mathfrak{C} \in \mathcal{E}(\mathfrak{P})$, while $s_{\mathfrak{C}}: \mathfrak{P} \rightrightarrows \mathfrak{C}$ and $r_{\mathfrak{C}}: \mathfrak{X} \rightrightarrows \mathfrak{C}$ are retractions, then

$$r_{\mathfrak{C}} = s_{\mathfrak{C}} r_{\mathfrak{P}}. \tag{2}$$

There may be retractions from $\mathfrak{X}$ onto $\mathfrak{P}$ defined in other ways; however, the notation $r_{\mathfrak{P}}$ will always be used to indicate a retraction $\theta_{\mathfrak{P}} r_{\mathfrak{A}}$ defined as above, hence $r_{\mathfrak{P}}$ is unique modulo the factor $\theta_{\mathfrak{P}}$.

In this connection notice that if α is the maximum of the diameters of the components of $\mathfrak{X}-\mathfrak{A}$, and β is the maximum of the diameters of $\mathfrak{E}$ for $\mathcal{E}(\mathfrak{P})\not\ni\mathfrak{E}\in\mathcal{E}(\mathfrak{A})$, then

$$(3) \qquad \rho\{\mathfrak{x}, r_{\mathfrak{P}}(\mathfrak{x})\} \leqq \alpha+\beta.$$

If $f\in F$ and lm is a Peanian factorization of f with middle space $\mathfrak{X}$ while $\mathfrak{P}\in\mathcal{P}(\mathfrak{X})$, then $f_{\mathfrak{P}}=lr_{\mathfrak{P}}m$ is a mapping from X into Y having a Peanian factorization $l(r_{\mathfrak{P}}m)$ with middle space $\mathfrak{P}$. In other words, $f_{\mathfrak{P}}\in F$. Different selections of $r_{\mathfrak{P}}$ will produce different mappings $f_{\mathfrak{P}}$; however, if $\mathfrak{E}\in\mathcal{E}(\mathfrak{P})$ then in view of (2) it is true that $f_{\mathfrak{E}}\equiv lr_{\mathfrak{E}}m$ $=ls_{\mathfrak{E}}r_{\mathfrak{P}}m\equiv(f_{\mathfrak{P}})_{\mathfrak{E}}\equiv f_{\mathfrak{P}\mathfrak{E}}$. Hence $f_{\mathfrak{P}\mathfrak{E}}$ is independent of the retraction $r_{\mathfrak{P}}$ in spite of the fact that $f_{\mathfrak{P}}$ is not.

II. If $f\in F$ has a Peanian factorization lm with middle space $\mathfrak{X}$, while $\mathfrak{A}_i\in\mathcal{A}(\mathfrak{X})$, $i=1, 2$, $\mathfrak{A}_1\cap\mathfrak{A}_2$ is a point, and $\mathfrak{A}_1\cup\mathfrak{A}_2=\mathfrak{X}$, then $\gamma(f_{\mathfrak{A}_1})+\gamma(f_{\mathfrak{A}_2})=\gamma(f)$.

Proof. If the theorem is false suppose that the sum on the left above is $\eta\neq\gamma(f)$, and select neighborhoods U and U_0 for η and $\gamma(f)$, respectively, such that

$$(4) \qquad U\cap U_0=\emptyset.$$

Let U_i be a neighborhood of $\gamma(f_{\mathfrak{A}_i})$, $i=1, 2$, such that

$$U_1+U_2\subset U.$$

Note that $\mathcal{E}(\mathfrak{A}_1)\cup\mathcal{E}(\mathfrak{A}_2)=\mathcal{E}(\mathfrak{X})$, and $\mathcal{E}(\mathfrak{A}_1)\cap\mathcal{E}(\mathfrak{A}_2)=\emptyset$. Since $\mathcal{A}(\mathfrak{X})\subset\mathcal{P}(\mathfrak{X})$ and γ is cyclicly additive, there is a finite subcollection $\mathcal{J}(U_i)$ of $\mathcal{E}(\mathfrak{A}_i)$ such that if $\mathcal{J}_i$ is a finite subcollection of $\mathcal{E}(\mathfrak{A}_i)$ containing $\mathcal{J}(U_i)$, then (understanding that $\mathfrak{A}_0=\mathfrak{X}$)

$$U_i\ni\sum\gamma(f_{\mathfrak{E}}), \qquad \mathfrak{E}\in\mathcal{J}_i, \qquad i=0, 1, 2.$$

If $\mathcal{J}$ is any finite subcollection of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U_0)\cup\mathcal{J}(U_1)\cup\mathcal{J}(U_2)$ let $\mathcal{J}_i=\mathcal{J}\cap\mathcal{E}(\mathfrak{A}_i)$, $i=1, 2$, and note that $\mathcal{J}_1\cap\mathcal{J}_2=\emptyset$ while $\mathcal{J}_1\cup\mathcal{J}_2=\mathcal{J}$. Hence

$$\begin{aligned} U_0 &\ni [\sum\gamma(f_{\mathfrak{E}}), \mathfrak{E}\in\mathcal{J}] \\ &= [\sum\gamma(f_{\mathfrak{E}}), \mathfrak{E}\in\mathcal{J}_1]+[\sum\gamma(f_{\mathfrak{E}}), \mathfrak{E}\in\mathcal{J}_2] \\ &\in U_1+U_2\subset U. \end{aligned}$$

Therefore $U\cap U_0\neq\emptyset$ in contradiction to (4).

II_a. The same as II except that the hypothesis is strengthened by the condition $\mathfrak{X}\in\mathcal{Q}(\mathfrak{X})$.

III. If $f\in F$ and has a Peanian factorization lm with middle space $\mathfrak{X}$, then for each neighborhood U of $\gamma(f)$ there is a $\mathfrak{Q}\in\mathcal{Q}(\mathfrak{X})$ such that $\mathfrak{Q}\subset\mathfrak{P}\in\mathcal{P}(\mathfrak{X})$ implies that $\gamma(f_{\mathfrak{P}})\in\mathfrak{U}$.

PROOF. There is a finite subcollection $\mathcal{J}(U)$ of $\mathcal{E}(\mathfrak{X})$ such that if $\mathcal{J}$ is any finite subcollection of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U)$, then $U \ni \sum \gamma(f_{\mathfrak{C}})$, $\mathfrak{C} \in \mathcal{J}$. The theorem will be proved if it can be shown that there is a $\mathfrak{Q} \in \mathfrak{L}(\mathfrak{X})$ such that $\mathcal{E}(\mathfrak{Q}) = \mathcal{J}(U)$.

If $\mathcal{J}(U) = \emptyset$ let $\mathfrak{Q}$ be any point in $\mathfrak{X}$. If $\mathcal{J}(U) \neq \emptyset$, suppose $\mathfrak{C}_1, \cdots, \mathfrak{C}_n$ are the elements in $\mathcal{J}(U)$. Define $\mathfrak{C}_1^* = \mathfrak{C}_1$ and join $\mathfrak{C}_2$ to $\mathfrak{C}_1^*$ by an arc $\mathfrak{K}_1$ with end points only in $\mathfrak{C}_1^* \cup \mathfrak{C}_2$. The arc $\mathfrak{K}_1$ may be a single point. Let $\mathfrak{C}_2^* = \mathfrak{C}_2 \cup \mathfrak{K}_1 \cup \bigcup \mathfrak{C}$ where the union is taken over the true cyclic elements $\mathfrak{C}$ in $\mathcal{J}(U)$ which have at least two points in common with $\mathfrak{K}_1$.

There is a first subscript n_1, if any, for which $\mathfrak{C}_{n_1}$ is not in $\mathfrak{C}_1^* \cup \mathfrak{C}_2^*$. Join $\mathfrak{C}_{n_1}$ to $\mathfrak{C}_1^* \cup \mathfrak{C}_2^*$ by an arc $\mathfrak{K}_2$ with end points only in $\mathfrak{C}_1^* \cup \mathfrak{C}_2^* \cup \mathfrak{C}_{n_1}$. Let $\mathfrak{C}_3^* = \mathfrak{C}_{n_1} \cup \mathfrak{K}_2 \cup \bigcup \mathfrak{C}$ where the union is taken over the true cyclic elements $\mathfrak{C}$ in $\mathcal{J}(U)$ which have at least two points in common with $\mathfrak{K}_2$.

This process stops after a finite number of steps m for want of a true cyclic element in $\mathcal{J}(U)$ not in $\mathfrak{C}_1^* \cup \cdots \cup \mathfrak{C}_m^* \equiv \mathfrak{Q}$. The Peano space $\mathfrak{Q}$ certainly has a cyclic chain approximation $\mathfrak{C}_1, \cdots, \mathfrak{C}_q$ such that $\mathfrak{C}_k$ is either an arc or $\mathfrak{C}_k \in \mathcal{J}(U) \subset \mathcal{E}(\mathfrak{X})$, $k = 1, \cdots, q$. Hence $\mathfrak{Q} \in \mathfrak{L}(\mathfrak{X})$ and $\mathcal{E}(\mathfrak{Q}) = \mathcal{J}(U)$.

III_a. If $f \in F$ and has a Peanian factorization lm with middle space $\mathfrak{X}$, then for each neighborhood U of $\gamma(f)$ there is a $\mathfrak{Q} \in \mathfrak{L}(\mathfrak{X})$ such that $\mathfrak{Q} \subset \mathfrak{R} \in \mathfrak{L}(\mathfrak{X})$ implies $\gamma(f_{\mathfrak{R}}) \in U$.

The question of the sufficiency of these conditions is considered next.

THEOREM. *If γ is a transformation from F into G having the properties* I_a, II_a *and* III_a, *then γ is cyclicly additive.*

PROOF. Let $f \in F$ and suppose lm is a Peanian factorization of f with middle space $\mathfrak{X}$. Suppose U is a neighborhood of $\gamma(f)$ and let $\mathfrak{Q}$ be the space in $\mathfrak{L}(\mathfrak{X})$ given by III_a. Define $\mathcal{J}(U) = \mathcal{E}(\mathfrak{Q})$, and suppose $\mathcal{J}$ is a finite subcollection of $\mathcal{E}(\mathfrak{X})$ containing $\mathcal{J}(U)$.

If $\mathcal{J} = \mathcal{J}(U)$ let $\mathfrak{R} = \mathfrak{Q}$. If $\mathcal{J} \neq \mathcal{J}(U)$, the construction used in the proof of III shows that there is an $\mathfrak{R} \in \mathfrak{L}(\mathfrak{X})$ such that $\mathcal{E}(\mathfrak{R}) = \mathcal{J}$ and $\mathfrak{R} \supset \mathfrak{Q}$. By III_a, $\gamma(f_{\mathfrak{R}}) \in U$.

But $\mathfrak{R}$ has a finite cyclic chain decomposition $\mathfrak{C}_1, \cdots, \mathfrak{C}_r$ such that either $\mathfrak{C}_k$ is an arc or $\mathfrak{C}_k \in \mathcal{E}(\mathfrak{X})$, $k = 1, \cdots, r$. Using the properties of a cyclic chain approximation, together with I_a and II_a it follows directly that

$$\begin{aligned} \gamma(f_{\mathfrak{R}}) &= \sum \gamma(f_{\mathfrak{R}\mathfrak{C}}), && \mathfrak{C} \in \mathcal{E}(\mathfrak{R}), \\ &= \sum \gamma(f_{\mathfrak{C}}), && \mathfrak{C} \in \mathcal{J}. \end{aligned}$$

Therefore,

$$U \ni \sum \gamma(f_{\mathfrak{E}}), \qquad \mathfrak{E} \in \mathcal{J},$$

and hence

$$\gamma(f) = \sum \gamma(f_{\mathfrak{E}}), \qquad \mathfrak{E} \in \mathcal{E}(\mathfrak{X}).$$

3. **An application.** Though greater generality is possible, suppose that X and Y are compacta, and for fixed $q \geqq 0$ consider the Čech cohomology groups $H^q(X)$ and $H^q(Y)$ where the coefficient group is discrete and does not change in the remainder of the discussion. The topology on $H^q(X)$ and $H^q(Y)$ is taken discrete and it should be mentioned that if $q=0$ one considers the reduced cohomology group.

If $f \in F$ then there is an induced homomorphism

$$f^*: H^q(Y) \rightarrow H^q(X).$$

Select any element $y \in H^q(Y)$, let $G = H^q(X)$, and define

$$\gamma(f) = f^*(y).$$

It will be shown that γ *is cyclicly additive*; *that is*, *if* lm *is a Peanian factorization of* $f \in F$ *with middle space* $\mathfrak{X}$, *then for each* $y \in H^q(Y)$,

$$f^*(y) = \sum f_{\mathfrak{E}}^*(y), \qquad \mathfrak{E} \in \mathcal{E}(\mathfrak{X}).$$

In view of the sufficiency theorem it is enough to check conditions $\mathrm{I_a}$, $\mathrm{II_a}$ and $\mathrm{III_a}$.

In the event $f \in F$ and has a Peanian factorization lm whose middle space is an arc, then f is clearly homotopic to a point and hence $f^*(y) = 0$.

To check $\mathrm{II_a}$ one will need the fact that: If X_1 and X_2 are closed subsets of X, and $X = X_1 \cup X_2$, while $f_i: X \rightarrow Y$ is a mapping for $i = 0, 1, 2$, such that $f_0 = f_i$ on X_i, $i = 1, 2$, and $x \in X_j$ implies $f_i(x) = y_0$, a single point of Y, $i, j = 1, 2$; $i \neq j$; then $f_0^* = f_1^* + f_2^*$. (See Borsuk [2].)

Suppose $\mathfrak{A}_i \in \mathcal{A}(\mathfrak{X})$, $i = 1, 2$, and $\mathfrak{A}_1 \cap \mathfrak{A}_2$ is a point $\mathfrak{x}_0$, while $\mathfrak{A}_1 \cup \mathfrak{A}_2 = \mathfrak{X}$. Define $X_i = m^{-1}(\mathfrak{A}_i)$ and $f_i = l r_{\mathfrak{A}_i} m$, $i = 1, 2$. Now X_1 and X_2 are closed subsets of X while $X_1 \cup X_2 = X$. Moreover, if $x \in X_i$, then $m(x) \in \mathfrak{A}_i$, hence $r_{\mathfrak{A}_i} m(x) = m(x)$ and so $f_i(x) = lm(x) = f(x)$; however, $f_j(x) = l r_{\mathfrak{A}_j} m(x) = l(\mathfrak{x}_0)$ since $r_{\mathfrak{A}_j}(\mathfrak{A}_i) = \mathfrak{x}_0$, $i, j = 1, 2$; $i \neq j$. Hence $f^* = f_1^* + f_2^*$ and $\gamma(f) = \gamma(f_{\mathfrak{A}_1}) + \gamma(f_{\mathfrak{A}_2})$, which shows that condition II and hence $\mathrm{II_a}$ is satisfied.

To check condition $\mathrm{III_a}$ recall that there is an $\epsilon(y)$ such that if $\phi_1: X \rightarrow Y$ and $\phi_2: X \rightarrow Y$ are two mappings with the property that $\rho\{\phi_1(x), \phi_2(x)\} < \epsilon(y)$, $x \in X$, then $\phi_1^*(y) = \phi_2^*(y)$. (See, for example, Hurewicz-Wallman [4, p. 140].)

Since l is continuous on a compact space there is a δ such that $\rho\{\mathfrak{x}_1, \mathfrak{x}_2\} < \delta$ implies that $\rho\{l(\mathfrak{x}_1), l(\mathfrak{x}_2)\} < \epsilon(y)$.

Let $\{\mathfrak{C}_k\}$ be a cyclic chain approximation to $\mathfrak{X}$, and select a so large that $n \geqq a$ implies that each component of $\mathfrak{X} - \mathfrak{A}_n$ has a diameter less than $\delta/2$, where $\mathfrak{A}_n = \bigcup \mathfrak{C}_k$, $k = 1, \cdots, n$. Consider $\mathfrak{A}_a$ and let $\mathfrak{K}_k$ be an arc whose end points are "end points" of $\mathfrak{C}_k$, $k = 1, \cdots, a$. Let $\mathfrak{E} \in \mathcal{E}$ if and only if $\mathfrak{E} \in \mathcal{E}(\mathfrak{A}_a)$, $d(\mathfrak{E}) < \delta/2$ and $\mathfrak{E} \cap \mathfrak{C}_k \neq \emptyset$ for exactly one integer $k = 1, \cdots, a$. Define $\mathfrak{Q} = [\mathfrak{A}_a - \bigcup \mathfrak{E}, \mathfrak{E} \in \mathcal{E}] \cup [\bigcup \mathfrak{K}_k, k = 1, \cdots, a]$.

It is easy to see that $\mathfrak{Q} \in \mathfrak{L}(\mathfrak{X})$, in fact $\mathfrak{A}_a$ is the smallest A-set containing $\mathfrak{Q}$, and if $\mathcal{E}(\mathfrak{Q}) \not\ni \mathfrak{E} \in \mathcal{E}(\mathfrak{A}_a)$, then $\mathfrak{E} \cap \mathfrak{Q}$ is an arc.

Now suppose that $\mathfrak{Q} \supset \mathfrak{R} \in \mathfrak{L}(\mathfrak{X})$. If $\mathfrak{A}$ is the smallest A-set containing $\mathfrak{R}$, then $\mathfrak{A} \supset \mathfrak{A}_a$ and hence each component of $\mathfrak{X} - \mathfrak{A}$ is in some component of $\mathfrak{X} - \mathfrak{A}_a$. Moreover, if $\mathcal{E}(\mathfrak{R}) \not\ni \mathfrak{E} \in \mathcal{E}(\mathfrak{A})$, then $\mathfrak{E} \in \mathcal{E}$ or $\mathfrak{E}$ is in some component of $\mathfrak{X} - \mathfrak{A}_a$. In either event $d(\mathfrak{E}) < \delta/2$. Hence by (3), $\rho\{x, r_{\mathfrak{R}}(x)\} < \delta$.

Since $f_{\mathfrak{R}} = l r_{\mathfrak{R}} m$, the selection of δ shows that $\rho\{f(x), f_{\mathfrak{R}}(x)\} < \epsilon(y)$, $x \in X$. Consequently $f^*(y) = f^*_{\mathfrak{R}}(y)$; that is, $\gamma(f) = \gamma(f_{\mathfrak{R}})$.

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Indiana University