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REMARKS ON THE NOTION OF RECURRENCE

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We give in several lines a simple proof of Poincaré's recurrence theorem.

THEOREM. *Let Ω be a point set of finite Lebesgue measure, and T a one-to-one measure-preserving transformation of Ω into itself.*[1] *Let $B \subset A \subset \Omega$ be measurable sets such that, if $b \in B$, $T^n b \notin A$ for all positive integral n. Then the measure $m(B)$ of B is 0.*

PROOF. First we show that, if $i < j$, $(T^i B)(T^j B) = 0$. Suppose $c \in T^j B$; then from the hypothesis on B it follows that j is the smallest integer such that $T^{-j} c \in A$. Hence $c \notin T^i B$. Now if $m(B) = \delta > 0$, Ω would contain infinitely many disjunct sets $T^n B$, each of measure δ. This contradiction proves the theorem.

The following generalization of the above theorem is trivially obvious: The result holds if we replace the hypothesis that T is measure-preserving by the following: If $m(D) > 0$, $\limsup_i m\{T^i(D)\} > 0$.

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[1] For a discussion in probability language see M. Kac, *On the notion of recurrence in discrete stochastic processes*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 1002–1010.

Another obvious generalization is this: Let C be the set of all points c of A such that $T^n c \in A$ for only finitely many n. Then $m(C)=0$ (for $C \subset \sum_{l=0}^{\infty} T^{-l}B$).

The following is a simple derivation of Kac's theorem on the mean recurrence time.[2]

THEOREM. *Let T above be metrically transitive. Let $a \in A-B$, and $n(a)$ be the smallest positive integer such that $T^n a \in A$. Let $m(A)>0$. Then*

$$\int_{A-B} n(a)dm = m(\Omega).$$

PROOF. Define $A_k = \{n(a)=k\}$. Let $i<j$, $i'<j'$, $j \neq j'$. We notice:

(a) $(T^i A_j)(T^{i'} A_{j'})=0$. For T has a single-valued inverse and $A_j A_{j'}=0$. If $T^i A_j$ and $T^{i'} A_{j'}$ had a point s in common, then $T^{-i}s \in A_j$, $T^{-i'}s \in A_{j'}$, in violation of the definition of j and j'.

(b) $$\int_{A-B} n(a)dm = m\left(\sum_{h=1}^{\infty} \sum_{l=0}^{h-1} T^l A_h\right).$$

(c) Metric transitivity implies that almost every point in Ω lies in some $T^l A_h$, that is, $m(\sum \sum T^l A_h)=m(\Omega)$.

This proves the desired result.

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[2] Kac, loc. cit. Theorem 2.