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A CONJECTURE OF KRISHNASWAMI

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Let $T(N)$ denote the number of right triangles whose perimeters do not exceed $2N$, and whose sides are relatively prime integers. A list of all such triangles whose perimeters do not exceed 10000 has been given by A. A. Krishnaswami.[1] On the basis of this table he conjectured that

$$T(N) \sim N/7. \tag{1}$$

The asymptotic formula

$$T(N) \sim \pi^{-2} N \log 4 \tag{2}$$

follows from the general theory of "totient points," as developed by D. N. Lehmer in 1900. A statement equivalent to (2) will be found in his paper[2] (p. 328).

The conjecture (1) is not far wrong since

$$\pi^2/\log 4 = 7.11941466.$$

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[1] A. A. Krishnaswami, *On isoperimetrical Pythagorean triangles*, Tôhoku Math. J. vol. 27 (1926) pp. 332–348. Two omissions in Table I may be noted: For $s=3450$, $a=50$, $b=19$; for $s=3465$, $a=55$, $b=8$. This table is the basis for the one at the end of the present paper.

[2] D. N. Lehmer, *Asymptotic evaluation of certain totient sums*, Amer. J. Math. vol. 22 (1900) pp. 293–335.

In this paper we give a short proof of the fact that

$$T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N). \tag{3}$$

The actual values of the error term for $N=500(500)5000$ are given in a short table at the end of this paper. The proof of (3) is based on the following lemmas.

LEMMA 1. *Let $\phi(m)$ denote the number of positive integers not exceeding m and prime to m. Then*

$$\Phi(x) \equiv \sum_{m\leq x} \phi(m) = 3\pi^{-2}x^2 + O(x \log x). \tag{4}$$

A proof of this well known result will be found for example in Hardy and Wright[3] (p. 266).

LEMMA 2. *Let $\Phi_e(x)$ and $\Phi_0(x)$ be defined by*

$$\Phi_e(x) = \sum_{m\leq x, m \text{ even}} \phi(m), \qquad \Phi_0(x) = \sum_{m\leq x, m \text{ odd}} \phi(m).$$

Then

$$\Phi_e(x) = \pi^{-2}x^2 + O(x \log x), \tag{5}$$

$$\Phi_0(x) = 2\pi^{-2}x^2 + O(x \log x). \tag{6}$$

PROOF. Since (6) follows from (4) and (5) it suffices to prove (5). To this effect we note that if m is even

$$\phi(m) = \begin{cases} \phi(m/2), & m \equiv 2 \pmod 4, \\ 2\phi(m/2), & m \equiv 0 \pmod 4. \end{cases} \tag{7}$$

Hence

$$\Phi_e(x) = \Phi_0(x/2) + 2\Phi_e(x/2) = \Phi(x/2) + \Phi_e(x/2).$$

Therefore

$$\Phi_e(x) = \sum_{\lambda=1}^{p} \Phi(2^{-\lambda}x) \qquad (p = [\log x/\log 2]).$$

Applying Lemma 1 we have

$$\Phi_e(x) = 3\pi^{-2}x^2\sum_{\lambda=1}^{p} 4^{-\lambda} + O(x \log x)$$

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[3] G. H. Hardy and E. M. Wright, *Introduction to the theory of numbers*, Oxford, 1938. Lemma 1 appears to be due to Mertens, Journal für Mathematik vol. 77 (1871) pp. 289–291.

$$= \pi^{-2}x^2 + O\left(x^2\int_p^\infty 4^{-t}dt\right) + O(x \log x).$$

Since $p > \log x/\log 4$, the integral is $O(1/x)$. Hence (5) follows.

LEMMA 3. *Let* $0<\theta<1$, *and define* $F(\theta, x)$, $F_e(\theta, x)$ *and* $F_0(\theta, x)$ *by*

$$F(\theta, x) = \sum_{\theta x < m \leqq x} m^{-2}\phi(m), \qquad F_e(\theta, x) = \sum_{\theta x < m \leqq x, m \text{ even}} m^{-2}\phi(m),$$

$$F_0(\theta, x) = \sum_{\theta x < m \leqq x, m \text{ odd}} m^{-2}\phi(m).$$

Then as $x \to \infty$, *with* θ *fixed,*

$$F(\theta, x) = -6\pi^{-2}\log\theta + O(x^{-1}\log x), \tag{8}$$

$$F_e(\theta, x) = -2\pi^{-2}\log\theta + O(x^{-1}\log x), \tag{9}$$

$$F_0(\theta, x) = -4\pi^{-2}\log\theta + O(x^{-1}\log x). \tag{10}$$

PROOF. Since (10) follows from (8) and (9) it suffices to prove (8) and (9). Now

$$\begin{aligned} F(\theta, x) &= \sum_{\theta x < m \leqq x} m^{-2}\phi(m) = \sum_{\theta x < m \leqq x} \{\Phi(m) - \Phi(m-1)\}m^{-2} \\ &= \sum_{\theta x < m \leqq x} \Phi(m)\{m^{-2} - (m+1)^{-2}\} \\ &\quad - \Phi(\theta x)[\theta x + 1]^{-2} + \Phi(x)[x+1]^{-2}. \end{aligned}$$

By Lemma 1 these last two terms cancel to some extent and together contribute only $O(x^{-1}\log x)$. As for the rest

$$\begin{aligned} &\sum_{\theta x < m \leqq x} \Phi(m)\{m^{-2} - (m+1)^{-2}\} \\ &\quad = 3\pi^{-2}\sum (1 - (1+m^{-1})^{-2}) + O\left(\sum m^{-1}(1-(1+m^{-1})^{-2})\log m\right) \\ &\quad = 3\pi^{-2}\sum 2m^{-1}(1+O(m^{-1})) + O\left(\sum m^{-2}\log m\right) \\ &\quad = 6\pi^{-2}\int_{\theta x}^x t^{-1}dt + O(x^{-1}) + O\left(\int_{\theta x}^x t^{-2}\log t\,dt\right) \\ &\quad = -6\pi^{-2}\log\theta + O(x^{-1}\log x), \end{aligned}$$

which gives (8). To prove (9) we note from (7) that

$$F_e(\theta, x) = F_0(\theta, x/2)/4 + F_e(\theta, x/2)/2 = F(\theta, x/2)/4 + F_e(\theta, x/2)/4.$$

Hence

$$F_e(\theta, x) = \sum_{\lambda=1}^{p} F(\theta, x/2^\lambda)4^{-\lambda} \qquad (p = [\log x/\log 2]).$$

Using (8) we find

$$F_e(\theta, x) = -6\pi^{-2}\log\theta\sum_{\lambda=1}^{\infty}4^{-\lambda} + O\left(\int_p^{\infty}4^{-t}dt\right) + O(x^{-1}\log x).$$

Since the integral is $O(x^{-1})$, (9) follows at once. This completes the proof of Lemma 3.

LEMMA 4. *Let $\phi(x, m)$ denote the number of integers $\leqq x$ and prime to m. Then*

$$|\phi(x, m) - xm^{-1}\phi(m)| < d(m)$$

where $d(m)$ is the number of divisors of m.

This follows easily from a familiar theorem of Legendre to the effect that

$$\phi(x, m) = \sum_{\delta | m}[x\delta^{-1}]\mu(\delta) \tag{11}$$

where μ is the Möbius function and the sum extends over all the divisors of m. In fact if we write

$$[x\delta^{-1}] = x\delta^{-1} - \epsilon(x, \delta)$$

so that

$$0 \leqq \epsilon(x, \delta) < 1,$$

then (11) becomes

$$\phi(x, m) = x\sum\delta^{-1}\mu(\delta) - \sum\epsilon(x, \delta)\mu(\delta).$$

The first sum is $m^{-1}\phi(m)$ and the second is less than

$$\sum_{\delta | m}1 = d(m)$$

in absolute value. This proves the lemma.

Finally we need one more lemma.

LEMMA. 5.

$$\sum_{m\leqq x}d(m) = O(x\log x).$$

This is a very weak corollary of a famous result of Dirichlet (see Hardy and Wright,[3] p. 262–263).

We are now in a position to prove the following theorem.

THEOREM. *Let $T(N)$ denote the number of integral right triangles whose perimeters do not exceed $2N$ and whose sides are relatively prime, then*

$$T(N) = \pi^{-2}N \log 4 + O(N^{1/2} \log N).$$

PROOF. It is well known that all integral right triangles (a, b, c) are given by the parametric equations

$$a = m^2 - n^2, \qquad b = 2mn, \qquad c = m^2 + n^2$$

where m, n are integers with

$$n \leqq m. \tag{12}$$

Since the perimeter is supposed not to exceed $2N$ we have

$$mn + m^2 \leqq N. \tag{13}$$

In order to avoid the cases in which a, b, c have a common factor it is necessary to suppose that we choose m, n so that

(14) m, n are coprime and not both odd.

$T(N)$ is then merely the number of pairs of positive integers (m, n) such that (12), (13) and (14) hold. In case $m \leqq (N/2)^{1/2}$, (13) is a consequence of (12). In case $(N/2)^{1/2} < m \leqq N^{1/2}$, (12) is a consequence of (13). Hence if we define

$$\psi(m) = \begin{cases} 1 & \text{if } m \leqq (N/2)^{1/2}, \\ m^{-2}N - 1 & \text{if } (N/2)^{1/2} < m \leqq N^{1/2} \end{cases} \tag{15}$$

then the number of integers n that go with a given m is the number of integers prime to m not exceeding $m\psi(m)$ or $m\psi(m)/2$ according as m is even or odd. Hence if we set $x = N^{1/2}$

$$T(N) = \sum_{m \leqq x, m \text{ even}} \phi(m\psi(m), m) + \sum_{m \leqq x, m \text{ odd}} \phi\left(\frac{m\psi(m)}{2}, m\right).$$

By Lemma 4,

$$\begin{aligned} T(N) &= \sum_{m \leqq x, m \text{ even}} \psi(m)\phi(m) + 2^{-1} \sum_{m \leqq x, m \text{ odd}} \psi(m)\phi(m) + R(N) \\ &= \sum\nolimits_1 + 2^{-1} \sum\nolimits_2 + R(N) \end{aligned} \tag{16}$$

where

$$|R(N)| \leqq \sum_{m \leqq x} d(m) = O(x \log x) = O(N^{1/2} \log N).$$

By (15) with $\theta = 2^{-1/2}$ we can write

$$\sum\nolimits_1 = \Phi_e(\theta x) + NF_e(\theta, x) - \Phi_e(x) + \Phi_e(\theta x),$$

$$\sum_2 = \Phi_0(\theta x) + NF_0(\theta, x) - \Phi_0(x) + \Phi_0(\theta x),$$

so that

$$\sum_1 + 2^{-1}\sum_2 = \Phi(\theta x) + \Phi_e(\theta x) - 2^{-1}\{\Phi(x) + \Phi_e(x)\} + 2^{-1}N\{F(\theta, x) + F_e(\theta, x)\}.$$

By Lemmas 1, 2, and 3 therefore we obtain after simplification

$$T(N) = \pi^{-2}(2\log 2)x^2 + O(x\log x) = \pi^{-2}(\log 4)N + O(N^{1/2}\log N).$$

The following small table illustrates the error in (3):

$$E(N) = T(N) - \pi^{-2}N\log 4.$$

The function $C(N)$ is defined by

$$C(N)N^{1/2}\log N = 10^3\,E(N)$$

and gives some idea of the possible constant implied by the O term of (3).

N	$T(N)$	ΔT	$\pi^{-2}N\log 4$	$E(N)$	$C(N)$
500	70	70	70.23049	−0.23049	−1.6596
1000	140	71	140.46099	−0.46099	−2.1103
1500	211	69	210.69148	+0.30852	+1.0893
2000	280	69	280.92197	−0.92197	−2.7123
2500	349	73	351.15246	−2.15246	−5.5022
3000	422	70	421.38296	−0.61704	−1.4071
3500	492	68	491.61345	+0.38655	+0.8007
4000	560	71	561.84394	−1.84394	−3.5152
4500	631	72	632.07444	−1.07444	−1.9041
5000	703		702.30493	+0.69507	+1.1541

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