THE COMPLETELY MONOTONIC CHARACTER OF THE MITTAG-LEFFLER FUNCTION $E_a(-x)$

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The Mittag-Leffler function is defined by the equation

$$E_a(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(ka + 1)} \cdot$$

A considerable literature is devoted to a study of the analytic character of this function. (See, for example, vol. 29 of Acta Mathematica.) Recently W. Feller communicated to me his discovery—by the methods of probability theory—that if $0 \leqq a \leqq 1$ the function $E_a(-x)$ is completely monotonic for $x \geqq 0$. This means that it can be written in the form

$$E_a(-x) = \int_0^{\infty} e^{-xt} dF_a(t),$$

where $F_a(t)$ is nondecreasing and bounded. In this note we shall prove this fact directly and determine the function $F_a(t)$ explicitly.

Since $E_0(-x) = 1/(1+x)$, $E_1(-x) = e^{-x}$ there is nothing to be proved in these cases. We assume then that $0 < a < 1$. By a standard representation[1]

$$E_a(-x) = \frac{1}{2\pi i a} \int_L \frac{e^{t^{1/a}}}{t + x} dt, \tag{1}$$

where L consists of three parts as follows:

C_1: *the line* $y = -(\tan \psi)x$ *from* $x = +\infty$ *to* $x = \rho$, $\rho > 0$.

C_2: *an arc of circle* $|z| = \rho \sec \psi$, $-\psi \leqq \arg z \leqq \psi$.

C_3: *the reflection of* C_1 *in the* x-axis.

We suppose $\pi > \psi/a > \pi/2$, while ρ is arbitrary but fixed.

In (1) replace $(x+t)^{-1}$ by $\int_0^\infty e^{-(x+t)u} du$. The resulting double integral converges absolutely, so that one can interchange the order of integration to obtain

$$E_a(-x) = \frac{1}{2\pi i a} \int_0^{\infty} e^{-xu} du \int_L e^{t^{1/a}} e^{-tu} dt.$$

It remains to compute the function

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[1] L. Bierberbach, *Lehrbuch der Funktionentheorie*, vol. 2, 1931, p. 273.

(2) $$F'_a(u) = \frac{1}{2\pi i a}\int_L e^{t^{1/a}} e^{-tu} dt,$$

and to prove it is non-negative when $u \geqq 0$. An integration by parts in (2) yields

$$F'_a(u) = \frac{1}{2\pi i a u}\int_L e^{-tu}\left(\frac{1}{a}\, t^{1/a-1}\right) e^{t^{1/a}} dt.$$

Now let $tu = z^a$. Then

(3) $$F'(u) = \frac{u^{-1-1/a}}{a}\left\{\frac{1}{2\pi i}\int_{L'} e^{-z^a} e^{zu^{1/a}}-dz\right\},$$

where L' is the image of L under the mapping.

Now consider the function

$$\phi_a(t) = \frac{1}{2\pi i}\int_{L'} e^{-z^a} e^{zt} dz.$$

This is known to be the inverse Laplace transform of

$$e^{-z^a} = \int_0^\infty e^{-xt}\phi_a(t) dt,$$

which is completely monotonic.[2] Hence

$$F'_a(u) = \frac{u^{-1-1/a}}{a}\phi_a(u^{-1/a}) \geqq 0.$$

From the explicit series[2] for $\phi_a(t)$ we find also that

(4) $$F'_a(u) = \frac{1}{\pi a}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sin \pi a k \Gamma(ak + 1) u^{k-1},$$

so that $F'_a(u)$ is an entire function.

It is of course possible to obtain (4) directly from (3). But a proof of its non-negative character without the intervention of the function e^{-x^a} eludes me.

It follows finally that $E_a(x)$ has no real zeros when $0 \leqq a \leqq 1$.

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[2] H. Pollard, *The representation of e^{-x^λ} as a Laplace integral*, Bull. Amer. Math. Soc. vol. 52 (1946) p. 908. The contour γ of that paper differs slightly from L', but is easily deformed into it.