# ON THE RELIABILITY OF THE MEMBRANE THEORY
## OF SHELLS OF REVOLUTION

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**1. Introduction.** The reliability of the membrane theory of shells is a somewhat controversial subject.[2] In my previous study of shells of revolution [MT] I have attempted to clarify the nature of the membrane theory as an approximate theory of elastic shells by deducing its differential equations as consequences of the three-dimensional infinitesimal theory of elasticity and of certain further assumptions, by discussing the type of boundary conditions to be used in problems concerning closed shells, and by proving the existence and uniqueness of solutions of the differential equations satisfying these boundary conditions. I developed also apparatus for quickly and efficiently finding the general solutions of the differential equations of the membrane theory for any given specific shell of revolution. Using this apparatus, in the present paper I shall show that:

1. In an open shell, or in a closed shell with a flat, sphere-like apex, the stress-resultants computed from the equations of the membrane theory will not exhibit a critical response to slight perturbations in the meridian curve, provided the curvature of the meridian curve is not changed very much.

2. In a closed shell with a pointed apex, a very slight change in the meridian curve in a very small region near the apex may entail very large changes in the stress resultants at all points of the shell, as computed from the membrane theory.

Both these results presuppose that the boundary condition at the apex is the "ring limit condition" stated at the end of §2. For a discussion of other possible boundary conditions, see §7.

These results and my previous treatment of the membrane theory show that in open shells or in closed shells with a flat, sphere-like apex, the stress resultants computed from the membrane theory

with the ring limit condition at the apex may be expected to be good approximations to the correct stress resultants, provided the support of the shell is consistent with a membrane state of stress,[3] but they cast doubt upon the reliability of the membrane theory in problems concerned with pointed shells.

2. **Fundamental apparatus for the subsequent discussion.** Let the meridian curve of the shell be $r = f(z)$, where the $z$-axis is the axis of revolution. Let $N_\phi$ and $N_\theta$ be the membrane stress resultants at a point in the directions of the meridian and the parallel curve respectively, and let $N_{\theta\phi}$ be the shear resultant. Let $X$, $Y$, $Z$ be the components of load per unit area in the directions of the parallel curve, meridian curve, and inward normal respectively. Let subscript $n$'s denote coefficients in complex Fourier series in the azimuth angle $\theta$. Then, as Neményi and I have shown,[4] the quantities $N_{\phi n}$, $N_{\theta n}$, and $N_{\theta\phi n}$ may be derived from the formulas

$$N_{\phi n} = \frac{(1 + f'^2)^{1/2}}{f^2} \mathfrak{U}_n, \qquad N_{\theta n} = \frac{f''}{(1 + f'^2)^{1/2}} \mathfrak{U}_n - f(1 + f'^2)^{1/2} Z_n,$$

(1)

$$in\ N_{\theta\phi n} = \frac{d}{dz}\left(\frac{\mathfrak{U}_n}{f}\right) + f(f'Z_n - Y_n),$$

where the stress functions $\mathfrak{U}_n(z)$ satisfy the differential equation

(2)
$$\frac{d^2\mathfrak{U}_n}{dz^2} + (n^2 - 1)\frac{f''(z)}{f(z)}\mathfrak{U}_n = g_n(z),$$

where

$$g_n(z) \equiv -f^2 f' \frac{dZ_n}{dz} + [(n^2 - 3)ff'^2 + n^2 f - f^2 f'']Z_n$$

(3)

$$+ f^2 \frac{dY_n}{dz} + 3ff'Y_n + \inf\ (1 + f'^2)^{1/2}X_n.$$

Terms in which $n = 0$ or $n = \pm 1$ are more conveniently treated with the aid of special explicit formulas [MT, pp. 128–129, 130–131] which avoid using the differential equation (2); in this investigation we shall limit our analysis to the terms in which $|n| \geq 2$, and our

[3] The normal stress only is prescribed at supports, it being supposed the support be such that the concomitant shearing stress is counterbalanced.

[4] P. Neményi and C. Truesdell, *A stress function for the membrane theory of shells of revolution*, Proc. Nat. Acad. Sci. U.S.A. vol. 29 (1943) pp. 159–162; [MT, pp. 126–127].

conclusions are therefore restricted to problems involving nonuniform load or support.

Let the base of the shell be at $z = 0$ and the apex at $z = z_0$. The most common boundary value problems involve given loads and a given mode of support. Then $N_{\phi n}(0)$, and hence $\mathfrak{U}_n(0)$, will be a prescribed quantity. If the shell has an open apex, then also $N_{\phi n}(z_0)$, and hence $\mathfrak{U}_n(z_0)$, will be a prescribed quantity. In the case of a shell with a closed apex, Flügge[5] has suggested that a proper boundary condition may be obtained by writing the equations of equilibrium of an annular section of the shell subtending a colatitude angle $\Delta\phi$, letting $\Delta\phi$ approach zero, and then letting the annular section approach the apex. This boundary condition we shall call the "ring limit condition." Let the apex of the shell be representable in the form

(4)         $f(z) = (z - z_0)^\mu g(z), \qquad \mu = 1/2 \text{ or } 1.$

Then the ring limit condition may be shown [MT, pp. 131–137] to take the form $\mathfrak{U}_n(z_0) = 0$, and solutions satisfying this condition will always exist. To follow the arguments of §5 it is essential to realize that the boundary conditions to be imposed on the solution of (2) must involve both the points $z = 0$ and $z = z_0$, rather than a single point.

3. **Preliminary observations.** Suppose we have two shells of revolution of nearly the same meridian curve loaded with the same load system and supported in the same way. From the differential equation (2) it is apparent that the difference between the two stress resultant systems will depend essentially on the difference between the two different ratios $f''/f$. In an open shell, or in portions of a closed shell which are far distant from its apex, we may study the effect of changing curvature very much while changing the shell radius very slightly. With this end in mind in §4 we shall show actually that an arbitrarily large change in the stress resultant distribution of any shell can be produced by introducing a sufficiently large change in the curvature of the meridian curve, at the same time keeping the shell radius arbitrarily close to its original value. This result is presented as of interest in itself, and is not offered as evidence of unreliability of the theory. Supposing, however, that both the change in curvature and the change in shell radius are kept small, so that the two meridian curves would seem hardly distinguishable, from the differential equation (2) we are led to expect singularities in the "complete" stress functions [MT, p. 128] $\mathfrak{U}_{nc}$ at the apex of a closed dome, where $f''/f$ usually [MT, p. 132] becomes infinite. In §5 we

[5] W. Flügge, *Statik und Dynamik der Schalen*, Berlin, 1934, p. 40.

shall show that if the two apexes are representable in the form (4) with exponent 1/2 (flat-topped shell), a reasonably good fit of the meridian curves will insure an excellent fit of the two values of $f''/f$ at the apex, but that if the exponent is 1 (pointed shell) the two meridian curves may appear to fit very closely with each other and yet the two values of $f''/f$ at the apex may be of entirely different orders of magnitude. We shall show also that a large difference in the two values of $f''/f$ in a very small region near the apex necessarily produces a large difference in the two stress resultant systems *throughout* the shells, casting doubt on the validity of the results of the membrane theory with the ring limit condition when it is applied to problems concerning pointed shells supported or loaded nonuniformly.

Before beginning the detailed analysis, however, let us mention the effect of three other types of special points in the meridian curve. First, a point of inflection in the meridian curve, while it produces an inflection in the complete stress functions $\mathfrak{U}_{nc}$, does not appear either from the equations (1), (2), and (3) or from two special cases treated in detail [MT, pp. 161–163] to cause any sort of singularity or noticeable variation in the stress-resultant system. A point where the tangent to the meridian curve is vertical, $f'=0$, seems to be equally neutral in its effect on the stress-resultant system. A point of horizontal tangency, where $f' = \infty$, will in general produce infinite stress-resultants according to the formulas (1), (2), and (3) unless $f=0$ at the same point. There appears to be no reason to doubt the validity of the membrane theory for shells whose meridian curves possess singularities of these types. The singularity at an apex, however, requires special analysis.

**4. The influence of curvature differences.** Preparatory to discussing the critical response of the membrane stress resultants to small changes in the curvature of the meridian curve, let us introduce a general superposition principle in the membrane theory of shells of revolution.

Suppose we have two shells whose meridian curves are $r = {}^1f(z)$ and $r = {}^2f(z)$. Let us superpose these two shells and their loads and obtain a third shell, $r = {}^1f(z) + {}^2f(z)$, loaded with the sum of the two original loadings. We shall compare the resulting membrane stress resultants with the sum of the two original stress resultants. We use left superscripts $c$, 1, and 2 to distinguish quantities associated with the combined, first, and second shells respectively. In particular, ${}^cf(z) \equiv {}^1f(z) + {}^2f(z)$. We define the excess stress functions

(5) $$\mathfrak{X}_n \equiv {}^c\mathfrak{U}_n - {}^1\mathfrak{U}_n - {}^2\mathfrak{U}_n.$$

Then from (2) and a straightforward calculation it can be shown that

(6) $$\mathfrak{X}_n'' + (n^2 - 1)\,({}^cf''/{}^cf)\,\mathfrak{X}_n = k_n,$$

where

(7) $$k_n \equiv (n^2 - 1)\left(\frac{{}^1f''\,{}^2f - {}^1f\,{}^2f''}{{}^cf}\right)\left(\frac{{}^1\mathfrak{U}_n}{{}^1f} - \frac{{}^2\mathfrak{U}_n}{{}^2f}\right) + {}^cg_n - {}^1g_n - {}^2g_n.$$

The stress resultant distribution derivable from the excess stress functions $\mathfrak{X}_n$ may be pictured as resulting from a fictitious load on the combined shell. The fictitious load is of two types: The first part of formula (7) represents the geometric change alone, and is independent of the original load systems, while the second part represents the different geometric resolution of the original load systems. In fact

(8) $$k_n \equiv -\, n^2\,{}^cf(1 + {}^cf'^2)^{1/2}P_n + {}^cg_n - {}^1g_n - {}^2g_n,$$

where

(9) $$P_n \equiv -\,\frac{n^2 - 1}{n^2}\cdot\frac{{}^1f''\,{}^2f - {}^1f\,{}^2f''}{({}^cf')^2(1 + {}^cf'^2)^{1/2}}\cdot\left(\frac{{}^1\mathfrak{U}_n}{{}^1f} - \frac{{}^2\mathfrak{U}_n}{{}^2f}\right).$$

$P_n$ is a fictitious load distribution in the direction perpendicular to the axis of revolution.

   Suppose, for example, we have a shell with meridian $r = {}^1f(z)$ subject to axially symmetric loading but nonuniformly supported: ${}^1g_n \equiv 0$. Let us superpose on it a small waviness, still keeping the apex of the shell closed:

(10) $$^2f \equiv \epsilon \sin k(z - z_0), \qquad {}^2g_n \equiv 0.$$

Then ${}^cg_n \equiv 0$ and [MT, p. 143]

(11) $$^2\mathfrak{U}_n = K_n\,\text{sh}\,(n^2 - 1)^{1/2}k(z - z_0);$$

hence by formula (9),

$$P_n = \frac{n^2 - 1}{n^2}$$

(12) $$\cdot\,\frac{({}^1f'' + k^2\,{}^1f)\epsilon \sin k(z - z_0)}{[{}^1f + \epsilon k \sin k(z - z_0)]^2(1 + [{}^1f' + \epsilon k \cos k(z - z_0)]^2)^{1/2}}$$

$$\cdot\left(\frac{K_n\,\text{sh}\,(n^2 - 1)^{1/2}k(z - z_0)}{\epsilon \sin k(z - z_0)} - \frac{{}^1\mathfrak{U}_n}{{}^1f}\right).$$

$^{1}\mathfrak{U}_n(z)$ and $^{1}f(z)$ are fixed functions, so that it is possible to choose $k$ sufficiently larger than their maximum values[6] so that

$$P_n \approx \frac{n^2 - 1}{n^2}$$

(13)

$$\cdot \frac{k^2 K_n \, {}^1f \, \text{sh} \, (n^2 - 1)^{1/2} k(z - z_0)}{[{}^1f + \epsilon \sin k(z - z_0)]^2 (1 + [{}^1f + \epsilon k \cos k(z - z_0)]^2)^{1/2}}.$$

We may now choose $\epsilon$ sufficiently small so that $\epsilon k$ is much smaller than the other magnitudes in formula (13), so that

(14)          $$P_n \approx \frac{n^2 - 1}{n^2} \cdot \frac{k^2 K_n \, \text{sh} \, (n^2 - 1)^{1/2} k(z - z_0)}{({}^1f)^2 (1 + {}^1f'^2)^{1/2}}.$$

Since initially we chose $k$ as large as we pleased it follows that $P_n$ may be made arbitrarily large even when $\epsilon$ is arbitrarily small, and that hence the membrane stress resultants derivable from the excess stress function will become arbitrarily large. This result is physically obvious, but has not been proved mathematically until now, so far as I know.

5. **Small perturbations of the meridian curve.** In §4 we set up apparatus by which we could discover large differences in stress resultant distributions due to a generally poor approximation of one shell by another, but those formulas are not convenient either for showing the absence of large differences when the approximation is close or for demonstrating large differences due to poor fit in the neighborhood of the apex alone. We now outline a method of estimating the difference in the stress resultant systems of two shells, approximately alike and loaded and supported in the same way.

We use the prefix $\Delta$ to indicate the difference between a quantity associated with the second shell with meridian curve $r = f(z) + \Delta f(z)$ and the corresponding quantity associated with the first one with meridian curve $r = f(z)$. Suppose that the two shells are of the same height $z_0$, and that if the apex is closed, each has the same exponent $\mu$ in the form (4). The same load distribution in the same geometric resolution is applied to each shell:

(15)          $$\Delta X = 0, \qquad \Delta Y = 0, \qquad \Delta Z = 0,$$

and the boundary conditions are the same for each:

(16)          $$\Delta N_{\phi n}(0) = 0, \quad \begin{cases} \Delta N_{\phi n}(z_0) = 0 & \text{for an open shell,} \\ \Delta \mathfrak{U}_n(z_0) = 0 & \text{for a closed shell.} \end{cases}$$

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[6] That finite maxima exist for these quantities follows from [MT, pp. 98, 132].

Then $\Delta\mathfrak{U}_n(z)$ will be that solution of the differential equation

$$(17) \quad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\left[\frac{f''}{f} + \Delta \frac{f''}{f}\right]\Delta\mathfrak{U}_n = \Delta g_n - (n^2 - 1)\mathfrak{U}_n\Delta\frac{f''}{f}$$

which assumes the values at $z=0$ and at $z=z_0$ given by the conditions (16) and (1). We may estimate the magnitude of $\Delta\mathfrak{U}_n$ in two steps.

I. First consider solutions of the equation

$$(18) \quad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\left[\frac{f''}{f} + \Delta \frac{f''}{f}\right]\Delta\mathfrak{U}_n = 0.$$

Since the solutions of differential equations (under suitable conditions) depend continuously upon the coefficients, as $\Delta(f''/f)\to 0$ the solutions of the equation (18) must approach certain solutions of the equation

$$(19) \quad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\frac{f''}{f}\Delta\mathfrak{U}_n = 0.$$

This equation is satisfied also by the complete stress functions $\mathfrak{U}_{nc}$ for the original shell, so its general solution we may write down in terms of functions we already know. We then attempt to estimate the maximum deviation of solutions of the equation (18) from these known functions.

II. We consider the equation

$$(20) \quad (\Delta\mathfrak{U}_n)'' + (n^2 - 1)\frac{f''}{f}\Delta\mathfrak{U}_n = \Delta g_n - (n^2 - 1)\mathfrak{U}_n\Delta\frac{f''}{f}.$$

Since we are presumed to know the complete primitive of the equation (19), and since the right-hand side of the equation (20) involves only known functions, we may write down the solution of equation (20) as a simple quadrature [MT, p. 127]. Using the result of step I, we may then estimate by how much this integral may deviate from the corresponding exact integral of the equation (17).

The prosecution of step II offers no difficulty whatever, but the result of step I is so complicated as to be useless. We shall set up the problem, however, because essential qualitative information may be gained from it. Suppose we have a function $Y(z)$ satisfying the differential equation

$$(21) \quad Y'' + (\phi + \epsilon)Y = 0$$

and assuming prescribed values at $z=0$ and $z=z_0$. By how much can

$Y(z)$ deviate from the solution of the differential equation

$$(22) \qquad y'' + \phi y = 0$$

which satisfies the same boundary conditions? We solve the equation (21) by iteration, using the solution of the equation (22) as the first approximating function:

$$Y_1 \equiv y$$

$$(23) \quad Y_{n+1} \equiv \left(1 - \frac{z}{z_0}\right) Y(0) + \frac{z}{z_0}\left[Y(z_0) + \int_0^{z_0} d\xi(z_0 - \xi)(\phi + \epsilon)Y_n\right]$$

$$- \int_0^z d\xi(z - \xi)(\phi + \epsilon)Y_n.$$

The bound for $|Y - y|$ is not simple, as it is in the familiar case illustrated in the text books,[7] first because of our two-point boundary condition, and second because the functions $\phi$ and $\epsilon$ in our case are $f''/f$ and $\Delta(f''/f)$ respectively, which are usually [MT, p. 132] singular at the apex of a closed dome. A bound exists, however, because it can be shown [MT, pp. 132–137] that the zero of $Y_n$ at $z = z_0$ is strong enough to keep $(z_0 - \xi)(\phi + \epsilon)Y_n$ bounded.

Now it can be shown [MT, p. 164] that $Y_0$, and hence $Y_n$, is of one sign for $0 \leq z \leq z_0$, providing $f''$ is of one sign. Then from the equation (23) we may see that even if $\epsilon \neq 0$ only in one small interval, the value of $Y_n$, and hence finally of $Y$, will be changed at every point except $z = 0$ and $z = z_0$ because of the presence of the integral from 0 to $z_0$, and further that if $\epsilon/\phi$ is large in this interval the change in $Y$ will be correspondingly large. Hence even if $\Delta(f''/f)/(f''/f)$ is large nowhere except in a small region near the apex, there will be a significant change in $\mathfrak{U}_n$ everywhere.

If now the apex is representable in the form (4) with $\mu$ equal to 1/2, and if $g(z)$ is imbedded in a family of continuous functions $g(z, b)$ of the parameter $b$ such that $g(z, 0) = g(z)$ and $f(z) + \Delta f(z) = (z - z_0)^{1/2}g(z, \Delta b)$, then it can be shown that

$$(24) \quad \frac{\Delta(f''/f)}{f''/f} = \frac{\dfrac{\partial}{\partial b}\left(\dfrac{g'}{g}\right)_{b=0} + (z - z_0)\dfrac{\partial}{\partial b}\left(\dfrac{g''}{g}\right)_{b=0}}{-1 + 4(z - z_0)\left(\dfrac{g'}{g}\right)_{b=0} + 4(z - z_0)^2\left(\dfrac{g''}{g}\right)_{b=0}}(z - z_0)\Delta b$$

$$+ O(\Delta b^2).$$

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[7] L. Bieberbach, *Theorie der Differentialgleichungen*, 3d ed., Berlin, 1930, pp. 39–41.

The term linear in $\Delta b$ vanishes when $z = z_0$, so that near the apex the effect of the function $\epsilon$ in equation (23) will be small, and consequently $|Y - y|$ will be small. If on the other hand the exponent in the form (4) is 1, and if we write $h(z, b)$ for $(z - z_0)g(z, b)$, we find that

$$(25) \qquad \frac{\Delta(f''/f)}{f''/f} = \left[\frac{\partial}{\partial b}\log h'' - \frac{\partial}{\partial b}\log h\right]_{b=0} \Delta b + O(\Delta b^2),$$

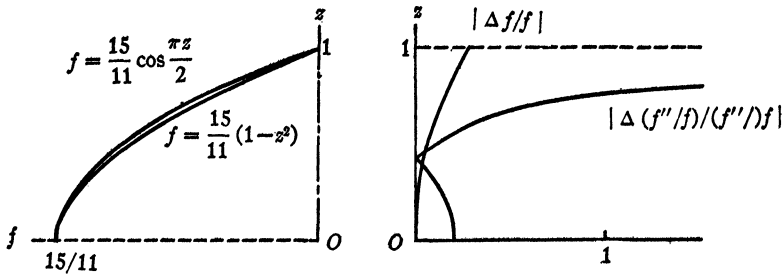and at the apex this quantity may be infinite. Consider, for example,



FIG. 1

the two shells previously treated in some detail by Flügge[8] and me [MT, pp. 155–158]:

$$(26) \qquad f \equiv A\cos\frac{\pi z}{2z_0}, \qquad f + \Delta f = A\left(1 - \frac{z^2}{z_0^2}\right).$$

These two meridian curves are those members of the family

$$(27) \qquad h(z, b) = A\left[bk\left(1 - \frac{z^2}{z_0^2}\right) + (1 - kb)\cos\frac{\pi z}{2z_0}\right]$$

which correspond to the values 0 and $1/k$ respectively for $b$. If $k$ is chosen very large, $\Delta b$ may be replaced by $b$ and the term $O(\Delta b^2)$ may be neglected in the formula (25), and we find that the value of that expression becomes infinite at the apex. In Figure 1 are plotted the two meridian curves and the exact values of $|\Delta f/f|$ and $|\Delta(f''/f)/(f''/f)|$. From the equation (27) it is possible to show that $\partial[\Delta(h''/h)/(f''/f)]/\partial b$ is infinite at the apex both when $b = 0$ and when $b = 1/k$. The value .273 approached by $\Delta f/f$ at the apex represents a local poorness of fit which should affect but little the correct stress resultants at points far distant from the apex; the two curves

[8] Op. cit. pp. 47–49.

are very closely matched, yet from equation (23) it is easy to see that the large values of $|\Delta(f''/f)/(f''/f)|$ in the upper part of the two shells will produce large differences everywhere in the stress resultants computed as solutions of the partial differential equations of the membrane theory.

6. **Evaluation of the membrane theory.** The formulas (17) and (23) show that a change in the quantity $f''/f$ anywhere along the meridian curve of a shell of revolution will affect the membrane stress resultants everywhere. Since at the apex of a closed shell that quantity usually becomes infinite, we reasonably expect large changes in it corresponding to small changes in the meridian curve. Formula (14) shows that if the apex is sphere like the ratio $\Delta(f''/f)/(f''/f)$ will vanish at the apex to the first order in the approximation parameter $\Delta b$ so that membrane stress resultants in shells with flat tops will not exhibit undue response to changes in the meridian curve. Formula (25) and the succeeding example show that for a pointed dome this ratio may become infinite, and hence that the stress resultant distribution in pointed shells may vary disproportionately in response to slight changes of the meridian curve.

We may summarize both the preceding results and my previous general discussion of the membrane theory [MT, pp. 108, 123]:

1. For an open shell or for a closed shell with a *flat, sphere-like apex* ($\mu = 1/2$), there will exist for a given middle surface and given shell loading a range of thicknesses for which the stress-resultant distribution computed from the membrane theory will be a correct first approximation to that computed from the stress distribution given by the three-dimensional theory, providing the load distribution is continuous and the support conditions are consistent with boundary conditions admissible in the membrane theory.[9] In this range slight local changes in the radius and in the curvature of the meridian curve will cause only slight changes in the stress resultants.

2. In a closed dome with a *pointed apex* ($\mu = 1$), a slight local change in the meridian curve of the apex may produce a very great change in the membrane stress resultants throughout the shell. Hence the membrane theory is not reliable in treating problems of nonuniform load and support for pointed domes.

It is possible to explain this unsatisfactory behavior of the membrane stress resultants in pointed shells. First, the apex of any pointed shell of any thickness is a point where the basic assumption of shell

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[9] Flügge, op. cit. pp. 119–120, makes an equivalent statement, drawn apparently from experience with examples, concerning shells of circular cylindrical form.

theory, namely "thickness/minimum radius of normal curvature $\ll 1$," is invalid. We are not interested in the apex for its own sake, but rather we wish to find formulas giving valid stress resultants at all points sufficiently far from it [MT, pp. 123–124]. Now examples of cylindrical shells given by Neményi[10] show that the effect of other types of singularities, such as discontinuities in the thickness or in the loads, is localized and insignificant at distant points if the stress and moment resultants are computed from the equations of the bending theory. Perhaps the apex of a pointed dome represents an equally local source of disturbance in the bending theory, but equation (23) shows that this singularity affects the stress resultant distribution computed from the equations of the membrane theory not only locally but throughout the shell.

   7. **Boundary conditions at the apex.** As mentioned in §1, we have employed the ring limit condition at the apex in the foregoing discussion. In the literature of shell theory it is customary to apply instead the more artificial requirement that the stress resultants remain finite at a closed apex. Flügge in proposing the ring limit condition claimed to prove that a membrane stress resultant system satisfying it would automatically remain finite,[11] but his analysis contains errors, and I showed that in the case of pointed shells the condition of finiteness could never be satisfied for terms in which $|n| \geqq 2$, but that the ring limit condition could always be satisfied by shells with meridian of form (4). Infinite stress resultants at the apex are in no way objectionable, since the apex is an idealization of no particular interest, behavior near which we are forced to consider only so as to obtain results valid for reasonable distances away. The present paper, however, shows that the membrane theory with the ring limit condition is not satisfactory for problems of pointed shells unsymmetrically loaded or supported.

   Now for pointed shells the annular section becomes undefined in the limit as the apex is approached [MT, p. 109]. Thus the ring limit condition is not physically natural. In re-proposing it originally I observed that it nevertheless led to a natural sort of boundary condition for solutions of the fundamental differential equation (2), that solutions satisfying it existed, and that it kept the membrane theory statically determinate for the same range of problems concerning pointed shells as for flat-topped shells.

   Nearly two years ago Professor Stoker suggested to me that finite-

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[10] Loc. cit.

[11] Op. cit. p. 40.

ness of the strain energy at the apex would be a physically more reasonable boundary condition, and I am in full agreement. There are two difficulties in the way of using this condition however.

The first one is mainly formal. The strain energy involves strains as well as stress resultants, so that to compute it we must solve the displacement equations as well as the equilibrium equations, the problem thus being no longer statically determinate. There seems to be no particular reason why two shells having identical loads and supports, the one having a flat apex and the other a pointed one, should lead to basically different boundary problems.

The second difficulty is much more formidable, however. To explain it, we must recall the derivation of the basic equations. In the customary presentation the strains are approximated by their values at the middle surface,[12] and the strain energy is a function of these strains. Now in my thesis[13] I showed that strains so approximated and employed in the usual macroscopic stress-strain relations[14] cannot satisfy the conditions of compatibility, even approximately. Hence I proposed a new derivation (see §8 below) of the fundamental differential equations [MT, pp. 102–108, 118–122], employing no approximate formulas for the strains and no macroscopic stress-strain relations; this derivation presupposes the existence of appropriate solutions of the full equations of the three-dimensional theory, including the conditions of compatibility, and from the microscopic stress-strain relations and the usual additional assumptions of shell theory (as listed in §8 below) deduces relations connecting the stress resultants and the displacements of the middle surface. The strain energy does not appear, and I do not know how to find a correct expression for it. Thus it is not clear how the condition of finiteness of the strain energy can be applied, even when general solutions of all the differential equations are known.

**8. A possible theory of average stresses.** A theory of shells giving results which are better indications of the correct three-dimensional stresses in problems concerned with pointed shells nonuniformly loaded or supported might be obtained by abandoning the stress resultants, which are defined, for example,

$$(28) \qquad N_\phi \equiv \int_{-\delta/2}^{\delta/2} \tau_{\phi\phi}\left(1 + \frac{x}{r_2}\right) dx,$$

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[12] Flügge, op. cit. pp. 51–52.
[13] Manuscript in Princeton Library.
[14] Flügge, op. cit. p. 50.

and returning to the averages originally introduced by Aron,[15] for example,

$$(29) \qquad\qquad T_\phi \equiv \int_{-\delta/2}^{\delta/2} \tau_{\phi\phi} dx.$$

At a sphere-like apex $r_2 \neq 0$, but at a pointed apex $r_2 = 0$ and hence $N_\phi$ must become infinite, even if $\tau_{\phi\phi}$ has no singularity at all. $T_\phi$, on the other hand, can never become infinite unless $\tau_{\phi\phi}$ becomes infinite and is thus a better guide to the behavior of the three-dimensional stress system. Now

$$(30) \qquad\qquad \left| \frac{N_\phi - T_\phi}{T_\phi} \right| \leqq \frac{\delta}{r_2} \cdot \frac{\displaystyle\int_{-\delta/2}^{\delta/2} \left| \tau_{\phi\phi} \right| dx}{\displaystyle\int_{-\delta/2}^{\delta/2} \tau_{\phi\phi} dx},$$

so that in regions far distant from the apex $N_\phi$ and $T_\phi$ differ by a quantity of the order neglected in the membrane theory. Hence at points far distant from the apex the differential equations of the average theory may be correctly approximated by those of the ordinary membrane theory, but near the apex the two systems of partial differential equations will be quite different because the difference $\delta/r_2$ becomes very large. The correct theory of averages[16] would then agree with the present membrane theory in cases when the latter gives correct results, but in problem of pointed domes non-uniformly loaded or supported it might avoid the membrane theory's unrealistic response to insignificant changes in the meridian curve.

It is not obvious, however, how a correct theory of averages could be formulated.

Since the equilibrium equations for the bending theory are exact

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[15] H. Aron, *Das Gleichgewicht und die Bewegung einer unendlich dünnen, beliebig gekrümmten elastischen Schale*, J. Reine Angew. Math. vol. 78 (1874) pp. 136–174.

[16] Some writers, for example, Timoshenko, *Theory of plates and shells*, New York and London, 1940, p. 352, make the error of initially neglecting the difference between averages and resultants, fancying they thereby simplify the derivation of the ordinary equations of the bending theory. There results a theory formally identical with the bending theory, but deduced with the aid of unnecessary approximations, for I have shown in [MT, pp. 118–122] that the equations of equilibrium of shells, expressed in terms of resultants, are exact consequences of the three-dimensional equilibrium equations, and that the displacement equations become poorer approximations to the correct displacement equations when averages are used in place of resultants. See also the observation of Flügge, op. cit. pp. 4–5.

and not approximate equations,[17] the best way to derive the equilibrium equations of the average theory would begin by expressing the stress and moment resultants in terms of the averages. If, for example,

$$(31) \qquad\qquad P_\phi \equiv - \int_{-\delta/2}^{\delta/2} x\tau_{\phi\phi}dx,$$

then from equations (28), (29), and (31) we have the relation

$$(32) \qquad\qquad N_\phi = T_\phi - (1/r_2)\, P_\phi.$$

In order to obtain exact expressions like this one for all 10 stress and moment resultants, it is necessary to use in all 13 different averages like those given by the definitions (29) and (31). By substituting all 10 of the relations of the type (32) into the equilibrium equations of the bending theory we obtain 5 exact equilibrium equations for the average theory.

The real difficulty lies in finding correct expressions giving the stress averages in terms of derivatives of the displacements of the middle surface. For the ordinary theory of shells I have pointed out a lucid way [MT, pp. 120–121] to deduce these relations: (I) to express the displacements of an arbitrary point in terms of the displacements of the projection of that point on the middle surface, then (II) to substitute these formulas into the expressions giving the three-dimensional strains in terms of the three-dimensional displacements, then (III) to substitute these values for the strains into Hooke's law and obtain the stresses, and finally (IV) to put the resulting values of the stresses into the definitions of type (28) and evaluate the integrals. Practical results are obtained in the bending theory by the aid of the characteristic assumptions of shell theory, namely[18]

A. $|\delta/R| \ll 1$, where $\delta$ is the thickness and $R$ the minimum radius of normal curvature of the middle surface.

B. $|\tau_{xx}/E| \ll |\epsilon_{xx} + \nu\Delta/(1-\nu)|$, where $E$ and $\nu$ denote Young's modulus and Poisson's ratio respectively, and $\Delta$ is the cubical dilatation.

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[17] See the preceding note. The equilibrium equations of the bending theory are therefore exact equilibrium equations in the strained coordinate system. To obtain a manageable theory we wish to be able to use these equations in the unstrained coordinates, so just as in three-dimensional elasticity we add the assumptions of very small displacements, so that the descriptions of the deformation given in the strained and unstrained coordinate systems will coalesce.

[18] [MT, pp. 104–105]. See also E. Reissner, *A new derivation of the equations for the deformation of elastic shells*, Amer. J. Math. vol. 63 (1941) pp. 177–184.

C. $\left|\epsilon_{x\phi}\right| \ll \delta/R, \left|\epsilon_{x\theta}\right| \ll \delta/R, \left|\epsilon_{xx}\right| \ll \delta/R, \left|\partial\epsilon_{xx}/\partial\phi\right| \ll 1, \left|\partial\epsilon_{xx}/\partial\theta\right| \ll 1.$

The assumptions A and C simplify the results of step I and hence of step II, the assumption B simplifies the result of step III, and the results of step IV become then a set of power series expansions in $\delta/R$ which are valid up to and including terms of 3rd order.

If the average theory is to be correct even when $r_2$ is arbitrarily small,[19] assumptions A and C must be modified. If one replaces $R$ by $r_1$ in these assumptions, they will then be correct for a range of problems dealing with thin shells of pointed apex, but unfortunately they no longer effect a simplification of step I sufficient to enable us to carry out the succeeding steps and arrive at manageable results.

The dominant characteristic of the ordinary membrane theory is that it is statically determinate: while assumptions A, B, and C are used in its derivation, the end result is a system of three differential equations involving as unknown functions only $N_\phi$, $N_\theta$, and $N_{\theta\phi}$ and making no mention of the displacements, which can be calculated from a second set of three partial differential equations as soon as the stress resultants are known. One estimates the orders of the stress and moment resultants from the relations expressing the stress-resultants in terms of the displacements of the middle surface, finding, for example, that $N_\phi = O(\delta/R)$, $M_\phi = O(\delta^3/R^3)$, so that a first order theory neglects bending moments. Since, as I have said above, it is not evident how to construct the two-dimensional stress-strain relations in the average theory when $r_2$ is allowed to be arbitrarily small, we do not have the apparatus for deciding whether or not there is a simple statically determinate theory of averages which will give a correct first approximation to the stress averages in pointed shells.

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[19] When $r_2 = 0$ the element of arc length in revolution coordinates [MT, p. 99] is no longer defined and the various differential formulas of elasticity therefore become indeterminate. Naturally we do not expect our results to be valid *at* the apex, but we wish our differential equations to be valid *near* it, that is, when $r_2$ is small but not actually zero.