# BOOK REVIEWS

*Length and area*. By Tibor Radó. American Mathematical Society Colloquium Publications, vol. 30. New York, American Mathematical Society, 1948. 6+572 pp. $6.75.

Since as long ago as the days of the early Greek mathematicians, men have been interested in the areas of surfaces. Special examples were studied, and with the invention of the calculus a formula was found expressing the area of a sufficiently smooth surface by means of a well known double integral. Nevertheless, the careful study of areas of continuous surfaces has been almost entirely a product of the preceding half-century. During this time about a dozen major definitions of area have been proposed, all being in agreement when applied to sufficiently well-behaved surfaces, but often having widely different properties when considered for all continuous surfaces. Of these various definitions, the one due to Lebesgue has been most thoroughly investigated, and upon it there now rests a theory possessing a high degree of completeness. It is this theory which constitutes the principal subject matter of Radó's *Length and area*. The "length" is of course an interesting subject; but its placidity, as contrasted with area, permits a very complete discussion in much less than half the book. The pattern of this discussion furnishes a model for developing the study of the area.

The theory presented in this book is the outstanding application of analytic topology to a problem in analysis. Since both the topology and the analysis are essential, the author devotes the first chapter to "background material." First the ideas of curve and surface are briefly discussed (a full discussion occurs later) and the distinction is drawn between definitions based on measure theory and those (like Lebesgue's) based on semi-continuity. Examples show that even in elementary situations there is need for care and precision. Next there are 17 pages of résumé of important theorems of topology, and 20 pages of résumé of analysis, particularly set-functions and integrals.

Part II is largely concerned with the topology of curves and surfaces. A transformation on a Peano space to a Peano space is monotone if the inverse image of each point is connected; it is light if the inverse image of each point is totally disconnected. The "factorization theorem" is demonstrated: if $T(P)=P^*$ is continuous, there is a monotone transformation $M(P)=\mathfrak{M}$ of $P$ onto a set $\mathfrak{M}$ and a light transformation $L(\mathfrak{M})=P^*$ such that $T=LM$. If the mappings $T_1(P_1)=P^*$, $T_2(P_2)=P^*$ are Fréchet equivalent they are also "K-equivalent," which means that $T_1$ and $T_2$ have simultaneous factorizations $T_1 = LM_1$, $T_2 = LM_2$ with the same middle-space $\mathfrak{M}$ and the same light factor $L$. In certain cases adequate for the sequel (such as when $P_1$, $P_2$ and $\mathfrak{M}$ are all spheres) $K$-equivalence implies Fréchet equivalence. Hereby the analysis is linked with the topology. The decomposition of $\mathfrak{M}$ into cyclic elements lets $T$ be studied as though analyzed into light transformations on spheres or two-cells or arcs or simple closed curves.

Part III begins with a study of interval functions (in two dimensions), their Burkill integrals and their extensions to functions of Borel sets. Next follows a chapter on functions (of one variable) of bounded variation, and absolutely continuous functions. These are used in developing the theory of the lengths of continuous curves, and of the concepts and consequences of convergence in length and in variation. Here again we see the excellent matching of the definitions of length, bounded variation, absolute continuity, Lebesgue integral and derivative.

In the next part we turn to the far more subtle theory of plane transformations and the substitution theory of double integrals. Of crucial importance is the concept of "essential multiplicity," developed by Radó from a beginning made by Geöcze. Let $T$: $z = t(w)$ map a Jordan region $R$ in the complex $w$-plane continuously into the $z$-plane. The multiplicity $N(z, T)$ of a point $z$ is the number of distinct points $w$ which map on $z$. Its *essential* multiplicity $\kappa(z, R)$ is the greatest number $k$ (possibly $\infty$) with the following property: if $k' < k$, for all continuous mappings $T'$ of $R$ which differ everywhere from $T$ by less than a certain positive $\epsilon = \epsilon(k')$ the multiplicity $N(z, T')$ exceeds $k'$. If $\mathcal{D}$ is an arbitrary domain, $\kappa(z, \mathcal{D})$ is defined as the limit of the essential multiplicities with respect to Jordan regions which expand and fill $\mathcal{D}$. The transformation $T$ is essentially of bounded variation (eBV) if $\kappa(z, \mathcal{D})$ is summable over the $z$-plane; a concept of essential absolute continuity (eAC) is also defined. We obtain a function of rectangles $r$ in $\mathcal{D}$ by mapping $r$ by $T$ into the $z$-plane and retaining only the image-points at which $\kappa(z, r) \neq 0$; the measure of this set is a function of $r$ which has a derivative $D_\kappa(w)$ for almost all $w$ in $\mathcal{D}$. This is, roughly, a "local area-magnification ratio," nonessential images being discarded. It is thus analogous to the absolute value of the Jacobian. A local topological index $i_e(w)$ of the mapping is defined, and the "essential generalized Jacobian" is defined to be $\mathfrak{J}_e(w) = i_e(w)D_\kappa(w)$. If the ordinary Jacobian exists almost everywhere, it is equal to $\mathfrak{J}_e$ almost everywhere. For all finite-valued measurable functions $H(z)$ the formula

$$\iint_{\mathfrak{D}} H(t(w)) \mid \mathfrak{J}_e(w) \mid = \iint H(z)\kappa(z, \mathfrak{D})$$

holds whenever either integral exists. Also, if $\nu(z)$ is the sum of $i_e(w)$ for all inverse images $w$ of $z$, then

$$\iint_{\mathfrak{D}} H(t(w))\mathfrak{J}_e(w) = \iint H(z)\nu(z)$$

if the left member exists.

The family of eAC transformations is not merely larger than the corresponding families in earlier theories of plane transformations; it also has closure properties which simplify the task of establishing that various special types of transformations are actually eAC.

The theory culminates in Part V, devoted to areas of surfaces. The strength of the results attained can be shown by quoting two theorems. The functions $x(u, v)$, $y(u, v)$, $z(u, v)$ $(0 \leqq u \leqq 1,\ 0 \leqq v \leqq 1)$ defining $S$ furnish three plane transformations, by projection. The square root of the sum of the squares of the three essential generalized Jacobians will be denoted by $W_e(u, v)$; if the ordinary Jacobians exist, the square root of the sum of their squares is $W(u, v)$. Then:

(A) If $A(S) < \infty$, the three plane transformations are eBV, and $W_e$ is defined almost everywhere in the unit square and is summable; and its integral is at most $A(S)$, being equal to $A(S)$ if and only if the three plane transformations are eAC.

(B) If $A(S) < \infty$ and $W(u, v)$ is defined almost everywhere in the unit square, it is summable, and its integral is at most $A(S)$, being equal to $A(S)$ if and only if the three plane transformations are eAC.

By his choice of methods of proof and by clarity of exposition, the author has provided a well-engineered road into a difficult territory. However, in the multitude of theorems a reader might well be puzzled about the interrelations, motivations and origins of the mathematical objects he encounters. The author has therefore provided a final chapter to each part except the first, in which he casts a backward look over the preceding chapters, coordinates their contents, and indicates directions in which further research is needed.

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*The advanced theory of statistics.* Vol. 2. By M. G. Kendall. London, Griffin, and Philadelphia, Lippincott. 1st ed., 1946, 2d ed., 1948. 8+521 pp. 50s.

The great treatise, of which the first volume was reviewed in Bull. Amer. Math. Soc. vol. 51 (1945) p. 214, is now complete. For the