THE EXTENSION OF A HOMEOMORPHISM DEFINED ON THE BOUNDARY OF A 2-MANIFOLD

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1. **Introduction.** Suppose that M and $\mathfrak{M}$ are homeomorphic 2-manifolds with boundaries B and $\mathfrak{B}$, respectively. Then B ($\mathfrak{B}$) is the union of a collection $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$), $n>0$, of Jordan curves which are disjoint in pairs. Suppose h is a homeomorphism from B onto $\mathfrak{B}$. (It may be assumed that $h(J_i)=\mathfrak{J}_i$, $i=1, \cdots, n$.) It is the purpose of this paper to investigate the possibility of *extending* the homeomorphism h so as to obtain a homeomorphism from M onto $\mathfrak{M}$.

It will be shown that, if M (and therefore $\mathfrak{M}$) is orientable, then h *cannot always* be extended unless $n=1$. (A necessary and sufficient condition for the extendability is given in Theorem 1.) If M (and therefore $\mathfrak{M}$) is non-orientable, then the extension is *always* possible—a result which, at first glance, may appear rather implausible.

These results are generalizations of the Schoenflies theorem [**2**, p. 324][2] and, astonishingly enough, do not appear to have been mentioned elsewhere. It is possible that they may serve as instruments in generalizing an extension theorem of Adkisson and MacLane [**1**] from a statement involving 2-spheres to one concerned with 2-manifolds. In any event, the theorems will be employed in the representation problem for Fréchet surfaces in a manner comparable to that by which a similar theorem was used to obtain a partial solution (Youngs [**4**]).

2. **The theorems.** Using the notation of the introduction, suppose that M is orientable. A *concordant* orientation of $(J_1, \cdots, J_n)$ consists of an orientation on each Jordan curve, $J_1, \cdots, J_n$, such that the orientation induced on M by the orientation on J_i is independent of $i=1, \cdots, n$; or, in other words, there is an orientation on M such that $J_1+\cdots+J_n$ (J_i regarded as a cycle, $i=1, \cdots, n$) is the algebraic boundary of M. Hence each concordant orientation of $(J_1, \cdots, J_n)$ determines an orientation on M; namely, the orientation induced by J_i for any $i=1, \cdots, n$. Conversely each orientation on M determines a concordant orientation of $(J_1, \cdots, J_n)$; the orientation on J_i being induced by the orientation on M, $i=1, \cdots, n$.

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[2] Numbers in brackets refer to the bibliography.

Thus there are two concordant orientations of $(J_1, \cdots, J_n)$; given one, the other is obtained by reversing the orientation on J_i, $i=1, \cdots, n$.

Now consider the homeomorphism h and select a concordant orientation of $(J_1, \cdots, J_n)$. Then J_i is oriented and $h|J_i$ (that is, h restricted to J_i) determines an orientation on $\mathfrak{J}_i$, $i=1, \cdots, n$. This selection of orientations may or may not be a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. If it is, then h is said to carry a concordant orientation of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. It is obvious that if h carries one of the two concordant orientations of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$, then it carries the other concordant orientation of $(J_1, \cdots, J_n)$ into the second concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$.

Now suppose that the homeomorphism h can be extended so as to obtain a homeomorphism $h^*: M \twoheadrightarrow \mathfrak{M}$. (The heavy arrow indicates that the mapping is from M onto $\mathfrak{M}$). Select an orientation on M and consider the concordant orientation of $(J_1, \cdots, J_n)$ determined by the orientation on M. The homeomorphism $h|J_i$ induces an orientation on $\mathfrak{J}_i$, $i=1, \cdots, n$, while the homeomorphism h^* induces an orientation on $\mathfrak{M}$. It follows that this orientation on $\mathfrak{M}$ induces an orientation on $\mathfrak{J}_i$ which is precisely that induced by $h|J_i$, $i=1, \cdots, n$. Consequently the orientation induced on $\mathfrak{J}_i$ by $h|J_i$, $i=1, \cdots, n$, yields a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. In other words, h takes a concordant orientation of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$. Thus half of the first theorem listed below has been proved.

THEOREM 1. *If M and $\mathfrak{M}$ are homeomorphic orientable 2-manifolds with boundaries $B=J_1\cup \cdots \cup J_n$ and $\mathfrak{B}=\mathfrak{J}_1, \cup \cdots \cup \mathfrak{J}_n$ respectively $(n>0)$, then a homeomorphism $h: B \twoheadrightarrow \mathfrak{B}$ can be extended to a homeomorphism $h^*: M \twoheadrightarrow \mathfrak{M}$ if, and only if, h carries a concordant orientation of $(J_1, \cdots, J_n)$ into a concordant orientation of $(\mathfrak{J}_1, \cdots, \mathfrak{J}_n)$.*

THEOREM 2. *If M and $\mathfrak{M}$ are homeomorphic non-orientable 2-manifolds with boundaries $B=J_1\cup \cdots \cup J_n$ and $\mathfrak{B}=\mathfrak{J}_1\cup \cdots \cup \mathfrak{J}_n$ respectively $(n>0)$, then a homeomorphism $h: B \twoheadrightarrow \mathfrak{B}$ can always be extended to a homeomorphism $h^*: M \twoheadrightarrow \mathfrak{M}$.*

PROOF OF THEOREM 1. The sufficiency of the condition needs to be established. Assuming that $h(J_i)=\mathfrak{J}_i$, $i=1, \cdots, n$, let M^* and $\mathfrak{M}^*$ be the closed orientable 2-manifolds obtained by adjoining 2-cells to the bounding curves $J_1, \cdots, J_n$ and $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$. These manifolds are obviously homeomorphic; suppose that their 1-dimensional Betti number is $2j$, $j\geqq 0$. By a suitable cutting of M^* ($\mathfrak{M}^*$) one obtains the fundamental polygon $P^*:AA^{-1}$ ($\mathfrak{P}^*:\mathfrak{A}\mathfrak{A}^{-1}$), if $j=0$, or $P^*:A_1B_1A_1^{-1}B_1^{-1}\cdots A_jB_jA_j^{-1}B_j^{-1}$ ($\mathfrak{P}^*:\mathfrak{A}_1\mathfrak{B}_1\mathfrak{A}_1^{-1}\mathfrak{B}_1^{-1}\cdots\mathfrak{A}_j\mathfrak{B}_j\mathfrak{A}_j^{-1}\mathfrak{B}_j^{-1}$), if $j>0$, and the Jordan curves $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$) are interior to P^* ($\mathfrak{P}^*$). (See Seifert-Trelfall [3, chap. VI].) Let J_{n+1} ($\mathfrak{J}_{n+1}$) be the Jordan curve boundary of P^* ($\mathfrak{P}^*$) and P' ($\mathfrak{P}'$) be the 2-manifold obtained from P^* ($\mathfrak{P}^*$) by omitting the interiors of the 2-cells bounded by $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$).

Select an orientation on P' and consider the induced orientations on $J_1, \cdots, J_{n+1}$. The mapping $h|J_i$ induces an orientation on $\mathfrak{J}_i$, $i=1, \cdots, n$. It follows from the hypothesis that the orientation on $\mathfrak{P}'$ induced by $\mathfrak{J}_i$ is independent of $i=1, \cdots, n$. Consider $\mathfrak{J}_{n+1}$ to be given the orientation induced by the above orientation on $\mathfrak{P}'$. It may be assumed that the order $\mathfrak{A}\mathfrak{A}^{-1}$, if $j=0$, or $\mathfrak{A}_1\mathfrak{B}_1\mathfrak{A}_1^{-1}\mathfrak{B}_1^{-1}\cdots\mathfrak{A}_j\mathfrak{B}_j\mathfrak{A}_j^{-1}\mathfrak{B}_j^{-1}$, if $j>0$, agrees with the orientation on $\mathfrak{J}_{n+1}$, and that the order AA^{-1}, if $j=0$, or $A_1B_1A_1^{-1}B_1^{-1}\cdots A_jB_jA_j^{-1}B_j^{-1}$, if $j>0$, agrees with the orientation on J_{n+1}.

If $j=0$ select the vertex v ($\mathfrak{v}$) which is the first point of A ($\mathfrak{A}$). If $j>0$ select the vertex v ($\mathfrak{v}$) which is the first point of A_1 ($\mathfrak{A}_1$). Let $x_i\in J_i$ and $\mathfrak{x}_i=h(x_i)\in\mathfrak{J}_i$, $i=1, \cdots, n$. It follows that there are arcs $Q_1, \cdots, Q_n$ ($\mathfrak{Q}_1, \cdots, \mathfrak{Q}_n$) from v to $x_1, \cdots, x_n$ ($\mathfrak{v}$ to $\mathfrak{x}_1, \cdots, \mathfrak{x}_n$) respectively, such that: 1°. $Q_i\cap Q_k=v$ ($\mathfrak{Q}_i\cap\mathfrak{Q}_k=\mathfrak{v}$), $i\neq k$; $i, k=1, \cdots, n$. 2°. If P' ($\mathfrak{P}'$) is cut along these arcs then one obtains the polygon $P:Q_1J_1Q_1^{-1}\cdots Q_nJ_nQ_n^{-1}AA^{-1}$ ($\mathfrak{P}:\mathfrak{Q}_1\mathfrak{J}_1\mathfrak{Q}_1^{-1}\cdots\mathfrak{Q}_n\mathfrak{J}_n\mathfrak{Q}_n^{-1}\mathfrak{A}\mathfrak{A}^{-1}$), if $j=0$, or $P:Q_1J_1Q_1^{-1}\cdots Q_nJ_nQ_n^{-1}A_1B_1A_1^{-1}B_1^{-1}\cdots A_jB_jA_j^{-1}B_j^{-1}$ ($\mathfrak{P}:\mathfrak{Q}_1\mathfrak{J}_1\mathfrak{Q}_1^{-1}\cdots\mathfrak{Q}_n\mathfrak{J}_n\mathfrak{Q}_n^{-1}A_1\mathfrak{B}_1\mathfrak{A}_1^{-1}\mathfrak{B}_1^{-1}\cdots\mathfrak{A}_j\mathfrak{B}_j\mathfrak{A}_j^{-1}\mathfrak{B}_j^{-1}$), if $j>0$. This is the fundamental polygon for M ($\mathfrak{M}$) and it is to be noted that the oriented boundary arcs $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$) are found in the order of increasing indices in the above array. It follows that the homeomorphism h carrying J_i onto $\mathfrak{J}_i$, $i=1, \cdots, n$, can be extended to a homeomorphism h from the boundary of P onto the boundary of $\mathfrak{P}$ in such a manner that if x and y are to be identified by the identification mapping which obtains M from P, then $h(x)$ and $h(y)$ are identified by the identification mapping which obtains $\mathfrak{M}$ from $\mathfrak{P}$.

Now by the Schoenflies theorem there is an extension h^* of h which maps P homeomorphically onto $\mathfrak{P}$. The homeomorphism h^* of the theorem is simply the above h^* considered as a mapping from M onto $\mathfrak{M}$.

PROOF OF THEOREM 2. Assuming that $h(J_i)=\mathfrak{J}_i$, $i=1, \cdots, n$, let M^* and $\mathfrak{M}^*$ be the closed non-orientable 2-manifolds obtained by adjoining 2-cells to the boundary curves $J_1, \cdots, J_n$ and $\mathfrak{J}_1, \cdots, \mathfrak{J}_n$. These manifolds are homeomorphic; suppose that their 1-dimensional Betti number is $(k-1)$, $k>0$. By a suitable cutting of M^* ($\mathfrak{M}^*$) one obtains the fundamental polygon $P^*: A_1A_1 \cdots A_kA_k$ ($\mathfrak{P}^*: \mathfrak{A}_1\mathfrak{A}_1 \cdots \mathfrak{A}_k\mathfrak{A}_k$) and the Jordan curves $J_1, \cdots, J_n$ ($\mathfrak{J}_1, \cdots, \mathfrak{J}_n$) are interior to P^* ($\mathfrak{P}^*$). (See Seifert-Trelfall [**3**, chap. VI].)

Consider a fixed orientation on P^*. This determines an orientation on J_i, and the homeomorphism $h|J_i$ induces an orientation on $\mathfrak{J}_i$, which, in turn, determines an orientation on $\mathfrak{P}^*$, $i=1, \cdots, n$. If this

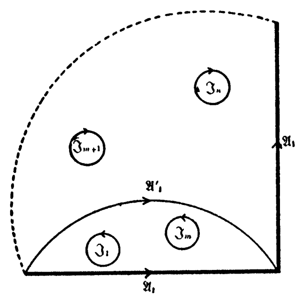

FIG. 1

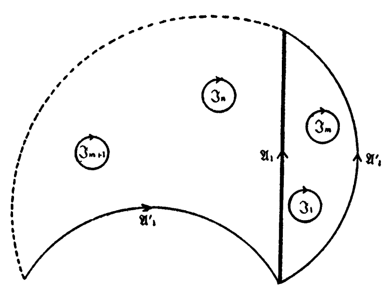

FIG. 2

orientation on $\mathfrak{P}^*$ is independent of $i=1, \cdots, n$, then it is readily seen that the proof can be completed as in Theorem 1. Suppose, therefore, that the orientations on $\mathfrak{J}_1, \cdots, \mathfrak{J}_m$ $(m<n)$ determine one orientation on $\mathfrak{P}^*$ while those on $\mathfrak{J}_{m+1}, \cdots, \mathfrak{J}_n$ determine the other. There is a cross cut $\mathfrak{A}'_1$ of $\mathfrak{P}^*$ joining the first point of $\mathfrak{A}_1$ to the last point of $\mathfrak{A}_1$ and separating $\mathfrak{J}_1 \cup \cdots \cup \mathfrak{J}_m$ from $\mathfrak{J}_{m+1} \cup \cdots \cup \mathfrak{J}_n$. (See Fig. 1.) Cut $\mathfrak{P}^*$ along $\mathfrak{A}'_1$ and identify the points of the two arcs labelled $\mathfrak{A}_1$ to obtain Fig. 2. Notice that in doing this one obtains the fundamental polygon $\mathfrak{P}': \mathfrak{A}'_1\mathfrak{A}'_1\mathfrak{A}_2\mathfrak{A}_2 \cdots \mathfrak{A}_k\mathfrak{A}_k$ of $\mathfrak{M}^*$ and $\mathfrak{J}_i$ now determines an orientation on $\mathfrak{P}'$ which is independent of $i=1, \cdots, n$. The proof is completed as in Theorem 1.

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