NOTE ON THE SPECTRAL REPRESENTATION OF A BOUNDED NORMAL MATRIX

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Every bounded normal (infinite) matrix N which possesses a bounded reciprocal may be written as a product of its polar factors as $N=PU=UP$, where P is positive definite and U is unitary. Using the corresponding spectral representations for P and U, a spectral representation of N may be obtained by means of Stieltjes integrals. Also, the cartesian factorization $N=H_1+iH_2$, where H_1 and H_2 are Hermitian and commutable, together with the spectral representations of H_1 and H_2 may be employed to obtain a spectral representation of N (cf. [1, 6][1] for references). It seems more natural to proceed directly by using the result on the moment problem for distribution functions in more than one dimension. The spectral representation of N is a consequence of the theorem below, which is the two-dimensional analogue of one employed by M. H. Martin [2] in obtaining the spectral representation of a bounded Hermitian matrix. The theorem holds in general for any finite number of bounded Hermitian matrices which commute with each other, but we consider the case of two such matrices for simplicity.

Notation. Let x, y denote one-column vectors with an infinity of (complex) components x_i, y_i, $i=1, 2, \cdots$. An infinite matrix A is said to be bounded if the least upper bound (l.u.b.) of the set of numbers $|y^*Ax|$ is finite ($y^*=\bar{y}'$, the conjugate transpose of y), where x and y range independently over the unit Hilbert sphere, that is, $|x|=(\sum_{i=1}^{\infty}|x_i|^2)^{1/2}=1$, $|y|=(\sum_{i=1}^{\infty}|y_i|^2)^{1/2}=1$. The matrix H is said to be Hermitian if $H=H^*$. For Hermitian matrices x^*Hx is real and

$$\underset{|x|=|y|=1}{\text{l.u.b.}} \; |y^*Hx| = \underset{|x|=1}{\text{l.u.b.}} \; |x^*Hx|.$$

If H is Hermitian and the greatest lower bound (g.l.b.) of the set of numbers x^*Hx is non-negative, that is,

$$\underset{|x|=1}{\text{g.l.b.}} \; x^*Hx \geqq 0,$$

then H is said to be non-negative definite. A positive definite matrix, P, is a Hermitian matrix satisfying

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[1] Numbers in brackets refer to references cited at the end of the paper.

$$\operatorname*{g.l.b.}_{|x|=1} x^*Px > 0.$$

We denote the unit matrix by E. A unitary matrix, U, is one for which $UU^* = U^*U = E$. N is said to be a normal matrix if $NN^* = N^*N$.

We first prove the following lemma.

LEMMA. *If P_1 and P_2 are two positive definite bounded matrices for which*

$$\text{(a)}\quad P_1P_2 = P_2P_1, \qquad \text{(b)}\quad \operatorname*{l.u.b.}_{|x|=1} x^*P_1x < 1, \quad \operatorname*{l.u.b.}_{|x|=1} x^*P_2x < 1$$

then there exist four bounded Hermitian matrices A_1, A_2, B_1, B_2 such that

(i) $A_1^2 = P_1$, $A_2^2 = P_2$, $B_1^2 = E - P_1$, $B_2^2 = E - P_2$,

(ii) *A_1, A_2, B_1, B_2 commute with each other.*

PROOF. Under the assumptions the existence of A_1, A_2, B_1, B_2 is guaranteed (cf. [**3**]). In fact, the "square roots" A_i and B_i may be represented as series

$$A_i = \sum_{n=0}^{\infty} c_n(E - P_i)^n, \qquad B_i = \sum_{n=0}^{\infty} c_nP_i^n, \qquad i = 1, 2,$$

where $c_0c_0 = 1$, $c_0c_1 + c_1c_0 = -1$,

$$\sum_{m=0}^{n} c_mc_{n-m} = 0, \qquad n = 2, 3, \cdots .$$

A_1 clearly commutes with B_1, and similarly for A_2 and B_2. By (a) A_1 and A_2 commute, as well as B_1 and B_2. Therefore A_1, A_2, B_1, B_2 all commute with each other.

THEOREM. *If H_1 and H_2 are bounded Hermitian matrices which commute and*

$$\mu_y(m, n) = y^*H_1^mH_2^ny, \qquad m, n = 0, 1, 2, \cdots, \; H_1^0 = H_2^0 = E,$$

where y is a fixed vector for which $|y| = 1$, then there exists an additive monotone set function $\phi = \phi_y = \phi_y(S)$ of the set S in the u, v-plane ($\phi(S) \equiv 0$ if S outside a certain bounded rectangle) such that

$$\mu_y(m, n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^mv^n\phi_y(dS^{uv}), \qquad m, n = 0, 1, 2, \cdots .$$

PROOF. There exist real numbers α_1, β_1, α_2, β_2 for which

$$(*) \qquad P_1 = \alpha_1H_1 + \beta_1E, \qquad P_2 = \alpha_2H_2 + \beta_2E \qquad (\alpha_1, \alpha_2 > 0)$$

satisfy the conditions of the lemma. Also, if a function ϕ exists for P_1 and P_2, one will exist for H_1 and H_2 since the transformation (*) only results in a translation and stretching of the spectrum of ϕ. Therefore we may assume H_1 and H_2 have the same properties as P_1 and P_2 of the lemma.

For a given double sequence of numbers $\mu(m, n)$ a necessary and sufficient condition that an additive monotone set function $\phi(S)$ exist with moments

$$\mu(m, n) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^m v^n \phi(dS^{uv}), \qquad m, n = 0, 1, 2, \cdots,$$

is that

$$\Delta_1^k \Delta_2^h \mu(m, n) \geqq 0, \qquad k, h, m, n = 0, 1, 2, \cdots,$$

where

$$\Delta_1^k \Delta_2^h \mu(m, n) = \sum_{r=0}^{k}\sum_{s=0}^{h} \binom{k}{r}\binom{h}{s}(-1)^{r+s}\mu(m+r, n+s)$$

(cf. [4] and [5]). For $\mu(m, n) = \mu_y(m, n) = y^* H_1^m H_2^n y$,

$$\begin{aligned}\Delta_1^k \Delta_2^h \mu_y(m, n) &= \sum_{r=0}^{k}\sum_{s=0}^{h} \binom{k}{r}\binom{h}{s}(-1)^{r+s}(y^* H_1^{m+r} H_2^{n+s} y)\\ &= y^*\left(\sum_{r=0}^{k}\sum_{s=0}^{h} \binom{k}{r}\binom{h}{s}(-1)^{r+s} H_1^{m+r} H_2^{n+s}\right)y\\ &= y^* H_1^m H_2^n (E - H_1)^k (E - H_2)^h y\end{aligned}$$

since H_1 and H_2 commute. Hence we must show that the matrix polynomial $H_1^m H_2^n (E-H_1)^k (E-H_2)^h$ is non-negative definite for $k, h, m, n = 0, 1, 2, \cdots$.

From the lemma we have the existence of A_1, A_2, B_1, B_2 and it follows that

$$\begin{aligned}H_1^m H_2^n (E - H_1)^k (E - H_2)^h &= (A_1^2)^m (A_2^2)^n (B_1^2)^k (B_2^2)^h\\ &= (A_1^m A_2^n B_1^k B_2^h)(A_1^m A_2^n B_1^k B_2^h)\\ &= (A_1^m A_2^n B_1^k B_2^h)(A_1^m A_2^n B_1^k B_2^h)^*.\end{aligned}$$

Since any bounded matrix of the form CC^* is non-negative definite, the theorem is proved.

If N is a normal bounded matrix, $N = H_1 + iH_2$ where $H_1 = (N+N^*)/2$, $H_2 = (N-N^*)/2i$ and H_1 and H_2 are both bounded Hermitian matrices which *commute* with each other since N is normal, that is, H_1 and H_2 satisfy the conditions of the theorem. Hence, we may write

$$\sum_{k=0}^{n}\binom{n}{k}(i)^{n-k}(y^*H_1^kH_2^{n-k}y)$$
$$= \sum_{k=0}^{n}\binom{n}{k}(i)^{n-k}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}u^kv^{n-k}\phi_y(dS^{uv}),$$

or

$$y^*(H_1 + iH_2)^n y = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(u + iv)^n\phi_y(dS^{uv});$$

but this is the same as

$$y^*N^n y = \int w^n\phi_y(dS^w)$$

where $w=u+iv$ and the integration is over the entire w-plane.

With proper normalizations $\phi_y(S)$ may be made unique.

References

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