ON AN INEQUALITY OF P. TURÁN CONCERNING LEGENDRE POLYNOMIALS

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The following remarkable inequality is due to the Hungarian mathematician P. Turán: If $P_n(x)$ denotes as usual Legendre's polynomial of the nth degree, we have

$$(1)\qquad \Delta_n(x) = (P_n(x))^2 - P_{n-1}(x)P_{n+1}(x) \geqq 0, \qquad n \geqq 1;\ -1 \leqq x \leqq 1,$$

with equality only for $x = \pm 1$. The purpose of this note is to give several proofs for this theorem different from that of Turán.[1]

1. Proof. The following arrangement is somewhat similar to that of Turán. By using the classical recursion

$$(2)\qquad P_{n+1}(x) = \frac{2n+1}{n+1}\, xP_n(x) - \frac{n}{n+1}\, P_{n-1}(x)$$

we find for the polynomial $\Delta_n(x)$ the representation

$$(3)\qquad P_n^2 + \frac{n}{n+1}\, P_{n-1}^2 - \frac{2n+1}{n+1}\, xP_nP_{n-1}.$$

This is a quadratic form in P_n and P_{n-1} which is positive provided

$$(4)\qquad \frac{n}{n+1} > \left(\frac{n+1/2}{n+1}\, x\right)^2, \qquad \text{or} \qquad |x| < \frac{(n(n+1))^{1/2}}{n+1/2} = \cos\theta_0.$$

For these x the theorem is already proved. For the remaining $x = \cos\theta$, that is, for $0 < \theta \leqq \theta_0$, we use Mehler's formula

$$(5)\qquad P_n(\cos\theta) = \frac{2}{\pi}\int_0^\theta \frac{\cos(n+1/2)u}{(2(\cos u - \cos\theta))^{1/2}}\, du$$

and obtain

$$(6)\qquad \begin{aligned} \Delta_n(\cos\theta) = \pi^{-2}\int_0^\theta\int_0^\theta &(\cos u - \cos\theta)^{-1/2}(\cos v - \cos\theta)^{-1/2} \\ &\cdot\{2\cos(n+1/2)u\cos(n+1/2)v - \cos(n-1/2)u\cos(n+3/2)v \\ &- \cos(n-1/2)v\cos(n+3/2)u\}\, dudv. \end{aligned}$$

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[1] I owe Mr. Turán also some other remarkable properties of the polynomial $\Delta_n(x)$.

The expression in the braces is

$$
\begin{aligned}
(7)\qquad & \cos (n+1/2)(u+v) + \cos (n+1/2)(u-v) \\
&\qquad - (1/2) \cos [(n-1/2)u + (n+3/2)v] \\
&\qquad - (1/2) \cos [(n-1/2)u - (n+3/2)v] \\
&\qquad - (1/2) \cos [(n-1/2)v + (n+3/2)u] \\
&\qquad - (1/2) \cos [(n-1/2)v - (n+3/2)u] \\
&\quad = \cos (n+1/2)(u+v)(1-\cos (u-v)) \\
&\qquad + \cos (n+1/2)(u-v)(1-\cos (u+v)),
\end{aligned}
$$

so that $\Delta_n > 0$ follows provided $(n+1/2)|u \pm v| \leqq (n+1/2)2\theta \leqq (n+1/2)2\theta_0 \leqq \pi/2$. But this is obvious since

$$
(8)\qquad \theta_0 < \frac{\pi}{2} \sin \theta_0 = \frac{\pi}{2} \frac{1}{2n+1} \cdot
$$

2. **Proof.** We expand $\Delta_n(x)$ in a finite series of Legendre polynomials which will contain only even terms:

$$
(9)\qquad \Delta_n(x) = c_0P_0(x) + c_1P_2(x) + c_2P_4(x) + \cdots + c_nP_{2n}(x).
$$

We show that the coefficients $c_1, c_2, \cdots, c_n$ are negative. Then the minimum will be reached if $P_m(x)$ is maximum, that is, for $x=1$. But $\Delta_n(1)=0$, thus the inequality will follow.

For this purpose we use a formula due to Adams, Ferrers and F. Neumann for the coefficients of the Legendre expansion of the product of two Legendre polynomials.[2] It is the simplest to state this formula in form of an integral:

$$
(10)\qquad
\begin{aligned}
i(a, b, c) &= \int_{-1}^{1} P_aP_bP_c dx \\
&= \begin{cases} 0 \text{ if } a+b+c \text{ odd}, \\ 0 \text{ if } a+b+c \text{ even but no triangle with sides } a, b, c \text{ exists}, \\ \dfrac{2}{2s+1}\, \dfrac{g_{s-a}g_{s-b}g_{s-c}}{g_s} \text{ if } a+b+c=2s,\ s \text{ integer, and a triangle with sides } a, b, c \text{ exists.} \end{cases}
\end{aligned}
$$

Here

[2] See, for instance, E. T. Whittaker and G. N. Watson, *Modern analysis*, American ed., 1943, p. 331.

$$g_s = \frac{1\cdot 3 \cdots (2s-1)}{2\cdot 4 \cdots 2s}; \qquad g_0 = 1. \tag{11}$$

In our case

$$\begin{aligned} \frac{2}{4\nu+1} c_\nu &= i(n, n, 2\nu) - i(n-1, n+1, 2\nu) \\ &= \frac{2}{2(n+\nu)+1}\left(\frac{g_\nu g_\nu g_{n-\nu}}{g_{n+\nu}} - \frac{g_{\nu-1} g_{\nu+1} g_{n-\nu}}{g_{n+\nu}}\right), \qquad \nu \geqq 1. \end{aligned} \tag{12}$$

Bu $g_\nu/g_{\nu-1}$ is increasing so that (12) is negative. This proves the assertion.

The same argument shows that in the expansion

$$(P_n(x))^2 - P_{n-l}(x)P_{n+l}(x) = c_0P_0(x) + c_1P_2(x) + \cdots + c_nP_{2n}(x) \tag{13}$$

the first l coefficients $c_0, c_1, \cdots, c_{l-1}$ are positive and all the others negative.

3. **Proof.** Professor Pólya has called my attention to the fact that inequalities of the Turán type occur in the study of entire functions of the form

$$\sum_{n=0}^{\infty} \frac{u_n}{n!} z^{n+r} = e^{-\alpha z^2+\beta z} z^r \prod (1+\beta_n z)e^{-\beta_n z} \tag{14}$$

where $\alpha \geqq 0$, β and β_n real, $\sum \beta_n^2$ convergent. These functions, studied by Laguerre, Pólya and others[3] are (apart from a constant factor) the only ones which are the limits of polynomials with only real roots. A sequence of such polynomials is for instance that of the Jensen polynomials

$$u_0 + \binom{n}{1} u_1 z + \binom{n}{2} u_2 z^2 + \cdots + u_n z^n \tag{15}$$

which approach the entire function (14) as $n \to \infty$, provided we replace z by z/n. Under the conditions mentioned these polynomials have only real roots. If we denote them by $z_1, z_2, \cdots, z_n$ the inequality $u_{n-1}^2 - u_{n-2}u_n \geqq 0$ is a trivial consequence of the following inequality:

[3] See G. Pólya and I. Schur, *Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen*, J. Reine Angew. Math. vol. 144 (1914) pp. 89–113; cf. pp. 96–97.

$$(16)\qquad \left(\frac{z_1 + z_2 + \cdots + z_n}{n}\right)^2 \geqq \frac{z_1z_2 + z_1z_3 + \cdots}{\binom{n}{2}}.$$

The generating function

$$(17)\qquad \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} z^n = e^{xz} J_0((1 - x^2)^{1/2} z)$$

shows, in view of well known properties of the Bessel function J_0, that the conditions mentioned above are indeed satisfied. This furnishes the theorem.

Taking into account the identities

$$(18)\qquad \begin{aligned} &\sum_{n=0}^{\infty} \frac{P_n^{(\lambda)}(x)}{P_n^{(\lambda)}(1)} \frac{z^n}{n!} = 2^{\lambda-1/2}\Gamma(\lambda + 1/2)e^{xz}((1 - x^2)^{1/2}z)^{1/2-\lambda} \\ &\qquad\qquad \cdot J_{\lambda-1/2}((1 - x^2)^{1/2}z), \qquad \lambda > -1/2, \\ &\sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \frac{z^n}{n!} = \Gamma(\alpha + 1)e^{z}(xz)^{-\alpha/2}J_\alpha(2(xz)^{1/2}), \qquad \alpha > -1, \\ &\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} z^n = e^{2xz-z^2} \end{aligned}$$

where $P_n^{(\lambda)}$, $L_n^{(\alpha)}$, H_n denote the ultraspherical, Laguerre and Hermite polynomials, respectively, we conclude the analogous inequalities for these polynomials.[4]

4. **Proof.** Finally we can avoid the use of transcendental functions and work only with the Jensen polynomial (15). It can be written in this particular case in the following form:

$$(19)\qquad \begin{aligned} &P_0(x) + \binom{n}{1}P_1(x)z + \binom{n}{2}P_2(x)z^2 + \cdots + P_n(x)z^n \\ &\qquad = (1 + 2xz + z^2)^{n/2}P_n\left(\frac{1 + xz}{(1 + 2xz + z^2)^{1/2}}\right). \end{aligned}$$

This identity can be shown by the ordinary generating function or by the generating function (17) or by the first integral of Laplace.[5] Now

[4] We follow the notation of G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloquium Publications, vol. 23, 1939, pp. 80, 98, 102. The inequality for the Hermite polynomials was pointed out to me by P. Turán.

[5] See G. Szegö, loc. cit. p. 87.

let x be fixed, $-1<x<1$. We obtain for the roots of the polynomial (19) in z the condition

$$\frac{1+xz}{(1+2xz+z^2)^{1/2}} = x_\nu \tag{20}$$

where x_ν denotes a root of P_n. Or

$$z = \frac{x(x_\nu^2-1) \pm x_\nu((1-x_\nu^2)(1-x^2))^{1/2}}{x^2-x_\nu^2}, \tag{21}$$

thus the roots in z are all real. Using the trivial inequality (16) the assertion follows.

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NOTE ON THE EIGENVALUES OF THE STURM-LIOUVILLE DIFFERENTIAL EQUATION

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In discussing eigenvalues and eigenfunctions of the Sturm-Liouville differential equation

$$L(u) + \lambda\rho u = 0, \qquad L(u) = (pu')' - qu,$$

with

$$\left.\begin{array}{l} p(x) \geqq m > 0 \\ q(x) \geqq 0 \\ \beta \geqq \rho(x) \geqq \alpha > 0 \end{array}\right\} \qquad \text{for } a \leqq x \leqq b, \text{ and for some } \alpha, \beta, \text{ and } m,$$

and the boundary conditions

$$u(a) = c_1u(b), \qquad u'(a) = c_2u'(b), \qquad c_1c_2p(a) = p(b),$$

we find that we can represent our eigenfunctions as unit normals in the directions of the principal axes of an ellipsoid in function space. We define our function space F as the set of all functions $v(x)$, $a \leqq x \leqq b$, which satisfy the boundary conditions of the Sturm-Liouville equation. The origin of our space will be the function $u(x)=0$. We can now metrize F by defining our inner product (u, v) for

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