LEMMA 6. *If Σ is recursive at x, then S is recursive at x.*

PROOF. Let U be an open neighborhood of x. By Lemma 4 there exists a compact set M in T such that $xM \subset U$ and $\Sigma \subset SM^{-1}$. Let V be a neighborhood of x for which $VM \subset U$. There exists an admissible set A such that $A \subset \Sigma$ and $xA \subset V$. Hence $xAM \subset U$. Define $B = S \cap AM$. Since $A \subset BM^{-1}$, B is an admissible set. Also $B \subset S$ and $xB \subset U$. The proof is completed.

The following theorem is an immediate consequence of Lemmas 5 and 6.

THEOREM. *If T is recursive at x, then S is recursive at x.*

An interpretation of admissibility arises if we define an admissible subset of T to be a relatively dense subset of T. The term "recursive" is then replaced by "almost periodic." For other applications, see the paper cited above.

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FIXED POINT THEOREMS FOR INTERIOR TRANSFORMATIONS

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If M is a bounded continuum in a Euclidean plane E which does not separate E and T is an interior continuous transformation of M onto a subset of E which contains M, does T leave a point of M invariant? It is the purpose of this paper to answer this question in the affirmative for certain types of locally connected continua.

Using a notation introduced by Eilenberg [2, p. 168][1] a continuum M will be said to have property (b) provided every continuous transformation of M into the unit circle S in the Cartesian plane, with center at o, is homotopic to a constant mapping, that is, a transformation which transforms each point of M into a single point of S. If T is a continuous transformation of a subset A of the plane E into a subset B of E, then for each point x of A let $T'(x)$ be the point y of S such that the directed line segment oy is parallel in direction and sense to the directed line segment $x, T(x)$. Then T' will be referred to as the transformation of A into S derived from T. Such a transformation

has often been considered in the study of fixed point properties of continua as for instance by Alexandroff and Hopf [1, p. 479]. If T is a continuous transformation of a subset A of E into a subset B of E, A will be said to have property (b') with respect to T if T leaves no point of A invariant and the transformation T' of A into S derived from T is homotopic to a constant mapping.

LEMMA 1. *There does not exist a bounded plane continuum M which has property* (b') *with respect to a continuous interior transformation T of M onto a topological 2-cell I which contains M.*

PROOF. Suppose the lemma is not true and that there does exist such a bounded plane continuum M. Without loss of generality we may take I to be the circular 2-cell in the Cartesian plane with center at o, radius 1, and boundary S. Let T' designate the transformation of M into S derived from T. By assumption, M has property (b') with respect to T and hence T' is homotopic to a constant mapping. By a theorem of Eilenberg [2, p. 168, Theorem 1'] there exists a continuous real valued function Φ which transforms M into a subset of R, the set of real numbers, such that for each point x of M, $T'(x)$ is the point $e^{i\Phi(x)}$. Since M is closed and bounded, $\Phi(M)$ is a bounded and closed subset of R. For each point x of I, let X designate the set consisting of all real numbers $\Phi(y)$, for y belonging to $T^{-1}(x)$, and let $\theta(x)$ be the greatest lower bound of the set X. Since T is an interior continuous transformation, it follows by a lemma of Eilenberg [2, p. 174] that T^{-1} is a continuous multi-valued transformation of I onto M, and hence that θ is a continuous transformation of I into R. For each point x of I let xz designate the directed half line with initial point at x which is parallel in direction and opposite in sense to the directed line segment oy, where y is the point $e^{i\theta(x)}$, and let $T_2(x)$ be the first point on xz distinct from x which xz has in common with S. We know $T_2(x)$ exists for each point x in I since M is a subset of I and since, from the continuity of θ and T^{-1} and the assumption that T leaves no point of M invariant, it follows that there is a point of M distinct from x lying on xz and hence a point of S distinct from x lying on xz. Thus, since θ is a continuous transformation of I into R, it follows that T_2 is a continuous transformation of I into the boundary of I which leaves no point of I invariant. But this contradicts the Brouwer fixed point theorem for 2-cells. The lemma is therefore true.

THEOREM 1. *If T is an interior continuous transformation of a locally connected unicoherent bounded plane continuum M onto a topological 2-cell which contains M, then T leaves a point of M invariant.*

PROOF. For suppose T leaves no point of M invariant. Then the continuous transformation T' of M into S derived from T exists and, by a theorem of Eilenberg [2, p. 168, Theorem 6'], since M is unicoherent and locally connected, M has property (b), and hence has property (b') with respect to T. But this contradicts Lemma 1.

COROLLARY. *If T is an interior continuous transformation of a bounded locally connected plane continuum which does not separate the plane onto a topological 2-cell which contains M, then T leaves a point of M invariant.*

PROOF. For a plane continuum which does not separate the plane is unicoherent.

THEOREM 2. *If T is an interior continuous transformation of a topological 2-cell I onto a continuum M which contains I, then T leaves a point of I invariant.*

PROOF. Suppose T leaves no point of I invariant. Then the derived transformation T' exists and, since I is unicoherent, it follows by the previously cited theorem of Eilenberg that I has property (b) and hence has property (b') with respect to T. But T transforms some subset N of I onto I. Then N has property (b') with respect to T. But this contradicts Lemma 1.

It is easy to construct an example to show that Theorem 2 would not be true if the condition that T be interior were omitted from the hypothesis of the theorem.

Applications of the theorems of this paper to the theory of functions of a complex variable are suggested by the fact that a function of a complex variable, analytic in a region R, defines an interior continuous transformation of R into a subset of the complex plane.

REFERENCES

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