A CHARACTERISTIC PROPERTY OF AFFINE COLLINEATIONS IN A SPACE OF K-SPREADS

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1. Introduction. In a recent paper[1] M. S. Knebelman has proved among other things that a necessary and sufficient condition which a mapping of an affinely connected space V_n upon itself shall satisfy in order that the covariant differentiation and the variation (the Lie derivative) of a tensor be interchangeable is that the mapping be an affine collineation. The present note deals with a similar problem in a space of K-spreads[2] by showing that the same condition is also characteristic of the isomorphic transformations.[3]

2. Affine collineations. Let

$$(1)\qquad \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta} + \Gamma^i_{jk}(x, p) p^j_\alpha p^k_\beta = 0 \qquad \left(p^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}\right)$$

be the partial differential equations of the K-spreads in an N-dimensional space, where $i, j, k, \cdots = 1, 2, \cdots, N$; $\alpha, \beta, \cdots = 1, 2, \cdots, K$. The integrability conditions are assumed to be satisfied, namely,

$$R^{i}_{\cdot jkl} p^j_\alpha p^k_\beta p^l_\gamma = 0,$$

where we have placed

$$(2)\qquad \begin{aligned} R^{i}_{\cdot jkl} = {} & \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \frac{\partial \Gamma^i_{jl}}{\partial x^k} - (\Gamma^i_{jk}\,|^\tau_m \Gamma^m_{nl} - \Gamma^i_{jl}\,|^\tau_m \Gamma^m_{nk}) p^n_\tau \\ & + \Gamma^i_{nl}\Gamma^n_{jk} - \Gamma^i_{nk}\Gamma^n_{jl}, \end{aligned}$$

and

$$A^{\cdots}_{\cdots}\,|^\sigma_l = \partial A^{\cdots}_{\cdots} / \partial p^l_\sigma.$$

The conditions satisfied by the functions $\xi^i(x)$ such that the infinitesimal transformation

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[1] M. S. Knebelman, *On the equations of motions in a Riemann space*, Bull. Amer. Math. Soc. vol. 51 (1945) pp. 682–685.

[2] J. Douglas, *Systems of K-dimensional manifolds in an N-dimensional space*, Math. Ann. vol. 105 (1931) pp. 707–733.

[3] E. T. Davies, *On the isomorphic transformations of a space of K-spreads*, J. London Math. Soc. vol. 18 (1943) pp. 100–107.

$$\bar{x}^i = x^i + \xi^i(x)\delta t \tag{3}$$

shall determine an affine collineation are known to be

$$\xi^m|_{hk} + R^m_{\cdot hkl}\xi^l + \Gamma^m_{hk}|^\sigma_n \xi^n|_\sigma = 0 \tag{4}$$

with $\xi^n|_\sigma = \xi^n|_k p^k_\sigma$.

For simplicity, let us consider a tensor $X^i_{\cdot j}$ which depends on the p^l_σ as well as the x^l. The covariant derivative of $X^i_{\cdot j}$ is defined by the equation

$$X^i_{\cdot j}|_k = \frac{\partial X^i_{\cdot j}}{\partial x^k} - X^i_{\cdot j}|^\alpha_m p^n_\alpha \Gamma^m_{nk} + X^h_{\cdot j}\Gamma^i_{hk} - X^i_{\cdot h}\Gamma^h_{jk}. \tag{5}$$

Denoting

$$X^i_{\cdot j}|_k|_l = X^i_{\cdot j}|_{kl}$$

we can readily show that

$$\begin{aligned} X^i_{\cdot j}|_k|^\sigma_l - X^i_{\cdot j}|^\sigma_l|_k &= (\delta^i_m X^h_{\cdot j} - \delta^h_j X^i_{\cdot m} - X^i_{\cdot j}|^\alpha_m p^h_\alpha)\Gamma^m_{hk}|^\sigma_l, \\ X^i_{\cdot j}|_{kl} - X^i_{\cdot j}|_{lk} &= (\delta^i_m X^h_{\cdot j} - \delta^h_j X^i_{\cdot m} - X^i_{\cdot j}|^\alpha_m p^h_\alpha)R^m_{\cdot hkl} \end{aligned} \tag{6}$$

with an evident generalization for any tensor.

3. **An extension of Knebelman's theorem.** We are in a position to generalize the result of Knebelman to the case of affine collineations in a space of K-spreads.

THEOREM. *A necessary and sufficient condition that a mapping of a space of K-spreads upon itself shall satisfy in order that the covariant differentiation of a tensor be interchangeable with the Lie derivative is that the mapping be an affine collineation of the space.*

To prove this, we have to recall the definition of the Lie derivative of the tensor $X^i_{\cdot j}$,

$$\Delta X^i_{\cdot j} = \lim_{\delta t \to 0} \frac{X^i_{\cdot j}(\bar{x}, \bar{p}) - \overline{X}^i_{\cdot j}(\bar{x}, \bar{p})}{\delta t}, \tag{7}$$

when the variables x^i are subjected to by (3).

It is readily shown that

$$\Delta X^i_{\cdot j} = X^i_{\cdot j}|_l \xi^l + X^i_{\cdot j}|^\sigma_l p^\gamma_\sigma \xi^l|_\gamma - X^r_{\cdot j}\xi^i|_r + X^i_{\cdot r}\xi^r|_j, \tag{8}$$

whence follows the relation

$$(\Delta X^{i}_{\cdot j})\,|^{\rho}_{k} - \Delta(X^{i}_{\cdot j}\,|^{\rho}_{k}) = 0. \tag{9}$$

That is, *the partial differentiation* $\partial/\partial p^{k}_{\rho}$ *of a tensor is always interchangeable with the Lie derivative.*

In virtue of (6), (8) and (9) we obtain

$$\begin{aligned}(\Delta X^{i}_{\cdot j})\,|_{k} - \Delta(X^{i}_{\cdot j}\,|_{k}) = {} & (X^{i}_{\cdot j}\,|^{\alpha}_{m} p^{h}_{\alpha} + \delta^{h}_{j} X^{i}_{\cdot m} - \delta^{i}_{m} X^{h}_{\cdot j}) \\ & \cdot (\xi^{m}\,|_{hk} + R^{m}_{\cdot hkl}\xi^{l} + \Gamma^{m}_{hk}\,|^{\sigma}_{n}\xi^{n}\,|_{\sigma}),\end{aligned} \tag{10}$$

which is equal to zero when, and only when, ξ^{i} are solutions of (4). Thus we have completed the proof.

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