ON THE SUM OF THE RELATIVE EXTREMA OF $|f(z)|$ ON THE UNIT CIRCLE

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Summary and introduction. Dr. Erdös has made the following conjecture: "Let $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$, with $z_\nu=r_\nu e^{i\theta_\nu}$, $r_\nu\leqq 1$, $f(z)\neq z^n$. Let $z_\mu=e^{i\theta_\mu}$ $(\mu=1, 2, \cdots k)$ be the points on the circumference of the unit circle, for which $|f(z)|$ is a relative extremum. Then $\sum_{\mu=1}^{k}|f(z_\mu)|\leqq 2^n$, and the equal sign applies only if $f(z)=(z-e^{i\theta_0})^n$." It will be proved that this is correct for large values of n, but incorrect for small ones.

With $z=e^{i\theta}$,

$$|f(z)|^2 = \prod_{\nu=1}^{n}|e^{i\theta}-r_\nu e^{i\theta_\nu}|^2 = \prod_{\nu=1}^{n}[1+r_\nu^2-2r_\nu\cos(\theta-\theta_\nu)],$$

or, with $t=\tan(\theta/2)$,

$$|f(z)|^2 \equiv F(t) = (1+t^2)^{-n}$$

$$\cdot\prod_{\nu=1}^{n}[(1+r_\nu^2+2r_\nu\cos\theta_\nu)t^2-4r_\nu\sin\theta_\nu t+(1+r_\nu^2-2r_\nu\cos\theta_\nu)],$$

$$F(t) = \frac{At^{2n}+\cdots}{(1+t^2)^n},$$

$$F'(t) = \frac{(1+t^2)(2nAt^{2n-1}+\cdots)-2nt(At^{2n}+\cdots)}{(1+t^2)^{n+1}} = \frac{Bt^{2n}+\cdots}{(1+t^2)^{n+1}}.$$

It may be assumed for the moment that $|f(e^{i\theta})|$ is not an extremum for $\theta=\pi$, that is, $t=\infty$; otherwise the coordinate system may be rotated. Thus it is seen that the number of relative extrema of $|f(z)|$ on the unit circle cannot exceed $2n$.

If $|f(e^{i\theta})|$ is less than $2^n/2n$ for every real value of θ, then $\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})|$ is obviously less than 2^n.

Assume now that $|f(z)|$ has a relative extremum at $z=1$, and that $|f(1)|\geqq 2^n/2n$. The proof then proceeds in the following steps: It will be shown, that for large values of n:

(1) All but $o(n)$ roots must be in a region R_1 of the unit circle close to $z=-1$.

(2) There is an arc T of the circumference of the unit circle, close to $z=+1$, such that the sum of all the extreme values of $|f(e^{i\theta})|$ with $e^{i\theta}$ not on T is $O(2^n/n)$.

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(3) The total number of extrema on T is $O(\log n)$.

(4) The sum of all the extreme values is not greater than 2^n.

PROOFS. (1) Divide the unit circle and its interior into 3 parts $R_1R_2R_3$, where R_1 is outside of and on a circle with radius $2-n^{-1/2}$ and center at 1, R_2 is inside of this circle, but outside of and on a circle with radius $2^{1/2}$ and center at 1, and R_3 is the rest.

The number of roots in R_2 plus R_3 must be *less than* $n^{3/4}$, otherwise $|f(1)| \leqq 2^{n-n^{3/4}} \cdot (2-n^{-1/2})^{n^{3/4}} = 2^n(1-1/2n^{1/2})^{n^{3/4}} < 2^n/e^{n^{1/4}/2} < 2^n/2n$, for n large enough. Similarly the number of roots in R_3 must be *less than* $3 \log n$.

For points $z = r \cdot e^{i\theta}$ in R_1, $\cos((\pi-\theta)/2) \geqq (2-n^{-1/2})/2 = 1-n^{-1/2}/2$. Consequently for such points, θ differs from π by less than $3 \cdot n^{-1/4}$, for n large enough.

(2) Let T be the arc of the circumference of the unit circle with the midpoint at $z=1$ and central angle $12 \cdot n^{-1/4}$. If $z=e^{i\theta}$ lies on the arc of the unit circle complementary to T then the distance from the point z to any point of R_1 is less than $2-n^{-1/2}$, so

$$|f(z)| < (2-n^{-1/2})^{n-n^{3/4}} \cdot 2^{n^{3/4}} = 2^n\left(1-\frac{1}{2n^{1/2}}\right)^{n-n^{3/4}}$$

$$= O\left(\frac{2^n}{e^{n^{1/4}}}\right) < \frac{2^n}{4n^2},$$

for n large enough. Therefore the sum of all the extreme values of $|f(e^{i\theta})|$ with $e^{i\theta}$ not on T is less than $2n \cdot 2^n/4n^2 = 2^n/2n$. Along T, $|t| = |\tan(\theta/2)| < 4 \cdot n^{-1/4}$.

(3) $|f(z)|^2 \equiv F(t) = \prod_{\nu=1}^{n}(1+r_\nu^2+2r_\nu \cos \theta_\nu) \cdot \prod_{\lambda=1}^{2n}(t-t_\lambda) \cdot (1+t^2)^{-n}$, where $t_{2\nu-1}$ and $t_{2\nu}$ are the roots of $(1+r_\nu^2+2r_\nu \cos\theta_\nu)t^2 - 4r_\nu \sin\theta_\nu \cdot t + (1+r_\nu^2-2r_\nu \cos\theta_\nu) = 0$. This equation has real coefficients, and its discriminant is $-4(1-r_\nu^2)^2 \leqq 0$. Therefore

$$|t_{2\nu-1}| = |t_{2\nu}| = \left(\frac{1+r_\nu^2-2r_\nu\cos\theta_\nu}{1+r_\nu^2+2r_\nu\cos\theta_\nu}\right)^{1/2} = \left|\frac{z_\nu-1}{z_\nu+1}\right|.$$

For z_ν in R_1, $|t_\lambda| > 1/3n^{-1/4} = n^{1/4}/3$. For z_ν in R_2, $|t_\lambda| \geqq 1$. The relative extrema of $|f(z)|$ for which $f(z)=0$ do not contribute to $\sum_{\mu=1}^{k}|f(z_\mu)|$. All the other extrema are characterized by

$$G(t) \equiv \frac{F'(t)}{F(t)} = \sum_{\lambda=1}^{2n}\frac{1}{t-t_\lambda} - \frac{2nt}{1+t^2} = 0.$$

To find an upper bound for the number of those extrema on T, consider $G(t)$ as a function of the complex variable t and integrate

$G'(t)/G(t)$ along a circle with center O and radius p, where p is a number between $1/4$ and $1/2$ such that for $|t| = p$, $|t-t_\lambda| > 1/50 \log n$. This is possible, because the number of roots z_ν in R_3 is less than $3 \log n$, and therefore the number of $|t_\lambda| < 1$ is less than $6 \log n$. Along this circle

$$\begin{aligned}\left|\sum_{\lambda=1}^{2n} \frac{1}{t - t_\lambda}\right| &\leqq \sum_{\lambda=1}^{2n} \frac{1}{|t - t_\lambda|} \\ &= \sum_{z_\nu \text{in} R_1} \frac{1}{|t - t_\lambda|} + \sum_{z_\nu \text{in} R_2} \frac{1}{|t - t_\lambda|} + \sum_{z_\nu \text{in} R_3} \frac{1}{|t - t_\lambda|} \\ &< 2n \frac{1}{n^{1/4}/3 - 1/2} + 2n^{3/4} \cdot \frac{1}{1 - 1/2} \\ &\quad + 6 \log n \cdot \frac{1}{1/50 \log n} = o(n),\end{aligned}$$

and

$$\left|\frac{2nt}{1 + t^2}\right| \geqq \frac{2n \cdot 1/4}{1 + 1/4} > \frac{n}{3}.$$

Therefore

$$|G(t)| = \left|\sum_{\lambda=1}^{2n} \frac{1}{t - t_\lambda} - \frac{2nt}{1 + t^2}\right| > \frac{n}{3} - o(n) > \frac{n}{4}$$

for n large enough,

$$\begin{aligned}|G'(t)| &= \left|\sum_{\lambda=1}^{2n} \frac{-1}{(t - t_\lambda)^2} - \frac{2n}{1 + t^2} + \frac{4nt^2}{(1 + t^2)^2}\right| \\ &< 2n \frac{1}{(n^{1/4}/3 - 1/2)^2} + 2n^{3/4} \cdot \frac{1}{(1 - 1/2)^2} \\ &\quad + 6 \log n \cdot \frac{1}{(1/50 \log n)^2} + \frac{2n}{1 - 1/4} + \frac{n}{(1 - 1/4)^2} < 5n\end{aligned}$$

for n large enough.

Thus $G'(t)/G(t) < 20$, and $|(1/2\pi i) \cdot \int_O G'(t)/G(t)dt| < (1/2\pi) \cdot 2\pi p \cdot 20 \leqq 10$. But $(1/2\pi i) \cdot \int_O G'(t)/G(t)dt =$ number of roots $-$ number of poles of $G(t)$, and the number of poles in this circle is not greater than $6 \log n$. Therefore the number of roots of $G(t)$, or the number of relative extrema of $|f(z)|$ in $|\tan (\theta/2)| < 1/4$ (and consequently the number of those on T), is less than $7 \cdot \log n$.

(4) If a circle K with radius $10\cdot n^{-1/4}$ around $z=1$ contains in its interior any roots z_ν, then for every point of T, $|f(z)|$ is less than $16\cdot 2^{n-1}/n^{1/4}$, and the sum of all the extrema on T is less than $7\cdot\log n\cdot 16\cdot 2^{n-1}/n^{1/4}<2^n/2$ for n large enough. The sum of all the extrema on the unit circle will be less than $2^n/2+2^n/2n<2^n$.

If K contains no roots in its interior, then for all the roots z_ν in R_3,

$$|t_\lambda| = \left|\frac{z_\nu - 1}{z_\nu + 1}\right| > \frac{10\cdot n^{-1/4}}{2} = 5n^{-1/4}.$$

For every point on T, $|t| = |\tan(\theta/2)|$ is less than $4\cdot n^{-1/4}$, and

$$\left|\sum_{\lambda=1}^{2n}\frac{-1}{(t-t_\lambda)^2}\right| \leqq \sum_{\lambda=1}^{2n}\frac{1}{|t-t_\lambda|^2} < 2n\cdot\frac{1}{(n^{1/4}/3 - 4n^{-1/4})^2}$$
$$+ 2n^{3/4}\cdot\frac{1}{(1-4n^{-1/4})^2} + \frac{6\log n}{(5n^{-1/4}-4n^{-1/4})^2} = o(n);$$

$$\frac{4nt^2}{(1+t^2)^2} = o(n),$$

and

$$G'(t) = \sum_{\lambda=1}^{2n}\frac{-1}{(t-t_\lambda)^2} - \frac{2n}{1+t^2} + \frac{4nt^2}{(1+t^2)^2} = -\frac{2n}{1+t^2} + o(n),$$

or: $G'(t)$ is negative everywhere on T. Consequently $G(t)$ can vanish not more than once on T, and $|f(z)|$ has on T no other extremum except at $z=1$. In this case, $\sum_{\mu=1}^{k}|f(z_\mu)| \leqq |f(1)|+2^n/2n$.

If R_2 or R_3 contains any roots z_ν, then $|f(1)|<2^{n-1}(2-n^{-1/2})$, and $\sum_{\mu=1}^{k}|f(z_\mu)|<2^n-2^{n-1}/n^{1/2}+2^{n-1}/n<2^n$.

If every root z_ν is in R_1, then $|z_\nu+1|<3\cdot n^{-1/4}$ for $\nu=1, 2, \cdots, n$. Call "q" the largest of the values $|z_\nu+1|$. Then $|\theta_\nu-\pi|\leqq\sin^{-1}q<2q$, for every value of ν. In this case, $|f(z)|$ increases steadily as θ changes from $\pi-2q$ to $6n^{-1/4}$, or from $\pi+2q$ to $2\pi-6n^{-1/4}$, and $|f(e^{i\theta})|$ can have no relative extremum except $|f(1)|$ outside of the arc $\pi-2q<\theta<\pi+2q$. Each extremum on this arc is less than $(3q)^n$, $|f(1)|$ is less than $2^{n-1}(4-q^2)^{1/2}<2^n-2^{n-3}q^2$, and

$$\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})| < 2^n - 2^{n-3}q^2 + 2n(3q)^n$$
$$= 2^n - 2^{n-3}q^2\cdot\left[1 - 36n\left(\frac{3q}{2}\right)^{n-2}\right].$$

Since $q<3n^{-1/4}$, the expression in the parenthesis is positive for n large enough, and $\sum_{\mu=1}^{k}|f(e^{i\theta_\mu})|<2^n$, unless $q=0$. But $q=0$ means that every $z_\nu=-1$, and $f(z)=(z+1)^n$.

The theorem has thus been proved for large values of n.

(5) For small values of n the theorem is incorrect. For example, if $f(z)=(z+1)\cdot(z-1/2)$, then $|f(e^{i\theta})|$ has relative extrema for $\theta=0$, π, arc cos (1/8), and the sum of the values of $|f(e^{i\theta})|$ at these places is $4.18>2^2$.

$f(z)=(z+1)^2\cdot(z-1/3)$ is another example in which $\sum|f(z_\mu)| =8.14>2^3$.

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ON LOCALLY SIMPLE CURVES

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1. **Introduction.** A continuous transformation $f(A)=B$ is *locally simple* provided that for each $x\in A$ there is a neighborhood U of x such that $U\cdot f^{-1}(y)$ contains at most one point for each $y\in B$. For the case where A is a circle and B is planar, such mappings have been studied recently by Morse and Heins [**1**][1] and used effectively in investigating meromorphic and other functions by topological methods.

In this paper topological characterizations will be obtained for those continua (compact connected metric spaces) which admit locally simple representations on the circle, that is, which are the image of the circle under some locally simple mapping.

It is clear that if A is compact and metric, a mapping (continuous) $f(A)=B$ is locally simple if and only if there exists an $e>0$ such that any subset of A of diameter not greater than e maps topologically onto its image under f or, equivalently, for any $y\in B$, any two distinct points of $f^{-1}(y)$ are at a distance greater than e apart.

A simple arc ab in a continuum M will be called *doubly extensible* in M provided M contains a simple arc a_1abb_1 such that every point of ab is an interior point of a_1abb_1. A continuum homeomorphic with the letter θ or with the figure 8 is called a *θ-curve* or an *8-curve* respectively; one homeomorphic with the sum of two disjoint circles plus a segment joining them with just an end point in each will be called a

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[1] Numbers in brackets refer to the bibliography at the end of the paper.