# ALGEBRAS AND THEIR SUBALGEBRAS

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There exist sets of postulates for Boolean algebras which consist entirely of universal sentences[1] in three variables. From this the following statement clearly follows: If an algebra has the property that every subalgebra generated by three elements is a Boolean algebra, then the algebra is itself a Boolean algebra.

The question arises, whether the above statement can be strengthened to read: If an algebra has the property that every subalgebra generated by two elements is a Boolean algebra, then the algebra is itself a Boolean algebra. This question can be answered in the negative by the following theorem:

THEOREM 1. *There exists an algebra* $\Gamma$ *which is not a Boolean algebra, but such that every subalgebra of* $\Gamma$ *with two generators is a Boolean algebra.*

PROOF. If $x$ and $y$ are any entities, then by "$\{x, y\}$" we shall mean the set whose only members are $x$ and $y$; and by "$\langle x, y\rangle$" we shall mean the ordered couple whose first member is $x$ and whose second member is $y$.

Let $H$ be the intersection of all sets $K$ which satisfy the following conditions:

(i) $2\in K$, $3\in K$, and $4\in K$;

(ii) If $x$ and $y$ are any distinct elements of $K$, neither of which belongs to the other, then $\{x, y\}\in K$.

Let $H_1$ be the class which contains all the members of $H$, and in addition the number 5. (Thus $H_1$ has just one more member than has $H$—namely 5.)

Let $G$ be the class of all ordered couples $\langle\alpha, \beta\rangle$ where $\alpha$ is any member of $H_1$, and $\beta$ is either 0 or 1.

We wish now to define a unary operation $-$, and two binary operations $+$ and $\cdot$, over the set $G$, in such a way that the system $\Gamma=\langle G, +, \cdot, -\rangle$ will not be a Boolean algebra, but every subalgebra of $\Gamma$ with two generators will be a Boolean algebra.

If $\langle\alpha, \beta\rangle$ is any member of $G$, then we set

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[1] By a universal sentence we mean here a sentential function without any bound variables ("for all $x$," "there exists an $x$." and so on)—or a sentence obtained from such a sentential function by prefixing universal quantifiers. We do not impose the condition that a Boolean algebra must contain at least two elements.

$$-\langle\alpha, \beta\rangle = \langle\alpha, 1-\beta\rangle,$$

where the minus sign in the right member denotes ordinary arithmetical subtraction.

If $\alpha$ and $\beta$ are distinct members of $H$ such that $\alpha \notin \beta$ and $\beta \notin \alpha$, then we set

$$\langle\alpha, 0\rangle + \langle\beta, 0\rangle = \langle\{\alpha, \beta\}, 0\rangle,$$
$$\langle\alpha, 1\rangle + \langle\beta, 1\rangle = \langle 5, 1\rangle,$$
$$\langle\alpha, 1\rangle + \langle\beta, 0\rangle = \langle\beta, 0\rangle + \langle\alpha, 1\rangle = \langle\alpha, 1\rangle.$$

If $\alpha$ and $\beta$ are distinct members of $H$ such that $\beta = \{\alpha, \gamma\}$ for some $\gamma$, then we set

$$\langle\alpha, 0\rangle + \langle\beta, 0\rangle = \langle\beta, 0\rangle + \langle\alpha, 0\rangle = \langle\beta, 0\rangle,$$
$$\langle\alpha, 1\rangle + \langle\beta, 1\rangle = \langle\beta, 1\rangle + \langle\alpha, 1\rangle = \langle\alpha, 1\rangle,$$
$$\langle\alpha, 1\rangle + \langle\beta, 0\rangle = \langle\beta, 0\rangle + \langle\alpha, 1\rangle = \langle 5, 1\rangle,$$
$$\langle\alpha, 0\rangle + \langle\beta, 1\rangle = \langle\beta, 1\rangle + \langle\alpha, 0\rangle = \langle\gamma, 1\rangle.$$

If $\alpha$ is any member of $H_1$, we set

$$\langle\alpha, 0\rangle + \langle\alpha, 0\rangle = \langle\alpha, 0\rangle,$$
$$\langle\alpha, 1\rangle + \langle\alpha, 1\rangle = \langle\alpha, 1\rangle,$$
$$\langle\alpha, 1\rangle + \langle\alpha, 0\rangle = \langle\alpha, 0\rangle + \langle\alpha, 1\rangle = \langle 5, 1\rangle.$$

If, finally, $\langle\alpha, \beta\rangle$ is any member of $G$, then we set

$$\langle\alpha, \beta\rangle + \langle 5, 0\rangle = \langle 5, 0\rangle + \langle\alpha, \beta\rangle = \langle\alpha, \beta\rangle,$$
$$\langle\alpha, \beta\rangle + \langle 5, 1\rangle = \langle 5, 1\rangle + \langle\alpha, \beta\rangle = \langle 5, 1\rangle.$$

We define the operation $\cdot$ by means of De Morgan's law, as follows:

$$x \cdot y = -(-x + -y).$$

From the way in which the system $\Gamma = \langle G, +, \cdot, -\rangle$ has been defined, it is easily seen that every subalgebra with two generators is a Boolean algebra. Thus, for instance, if $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$ are such that $\alpha_1 \neq \alpha_2$, $\alpha_1 \neq 5 \neq \alpha_2$, $\alpha_1 \notin \alpha_2$, and $\alpha_2 \notin \alpha_1$, then $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$ generate a Boolean algebra with the eight elements: $\langle\alpha_1, 0\rangle$, $\langle\alpha_2, 0\rangle$, $\langle\{\alpha_1, \alpha_2\}, 0\rangle$, $\langle 5, 0\rangle$, $\langle\alpha_1, 1\rangle$, $\langle\alpha_2, 1\rangle$, $\langle\{\alpha_1, \alpha_2\}, 1\rangle$, $\langle 5, 1\rangle$. In other cases, the Boolean algebra generated by $\langle\alpha_1, \beta_1\rangle$ and $\langle\alpha_2, \beta_2\rangle$ may have fewer than eight elements.

On the other hand, the system $\langle G, +, \cdot, -\rangle$ is not itself a

Boolean algebra, since it does not satisfy the associative law. Consider, for instance, the elements $\langle 2, 0\rangle$, $\langle 3, 0\rangle$, and $\langle 4, 0\rangle$; we have $(\langle 2, 0\rangle+\langle 3, 0\rangle)+\langle 4, 0\rangle=\langle\{\{2, 3\}, 4\}, 0\rangle$, while $\langle 2, 0\rangle+(\langle 3, 0\rangle+\langle 4, 0\rangle)=\langle\{2, \{3, 4\}\}, 0\rangle$, and $\{\{2, 3\}, 4\}$ is not the same as $\{2, \{3, 4\}\}$.

COROLLARY. *It is not possible to give a set of postulates for Boolean algebra where each postulate is a universal sentence containing only two variables.*

One can establish analogous theorems for a number of other kinds of algebras. Thus, for example, suppose we define the operation $\circ$ in terms of the operations of $\Gamma$ as follows:

$$x \circ y = (x\cdot -y) \quad (-x\cdot y).$$

Then it is easily shown that the system $\langle G, \circ\rangle$ is not a group, but that every subalgebra of $\langle G, \circ\rangle$ with two generators is an Abelian group. Thus it follows that one cannot give a set of postulates for groups (or for Abelian groups) where each postulate is a universal sentence with only two variables. A similar result follows for rings, since every subalgebra with two generators of the system $\langle G, \circ, \cdot\rangle$ is a ring, while $\langle G, \circ, \cdot\rangle$ is not itself a ring.

Since every Boolean algebra is a distributive lattice, and since every lattice satisfies the associative law for addition, we see that there exists a system $\Delta=\langle G, +, \cdot\rangle$, which is not a lattice, but such that every subalgebra of $\Delta$ with two generators is a distributive lattice. Thus it is not possible to give a set of postulates for lattices where each postulate is a universal sentence with only two variables. The same is clearly true also of distributive lattices, and of modular lattices.

Similar results are easily seen to hold, finally, for closure algebras, and for Brouwerian algebras.[2]

Our theorem also finds application in connection with the axiomatization of the sentential calculus. It is well known that every Boolean algebra can be regarded as a matrix for the classical (two-valued) sentential calculus; thus every subalgebra with two generators of the system $\Gamma$ of Theorem 1 is a matrix for the classical sentential calculus. Moreover, the formula

$$[p \vee (q \vee r)] \rightarrow [(p \vee q) \vee r]$$

[2] For definitions of these algebras, see J. C. C. McKinsey and Alfred Tarski, *The algebra of topology*, Ann. of Math. vol. 45 (1944) pp. 141–191, and *On closed elements in closure algebras*, Ann. of Math. vol. 47 (1946) pp. 122–162.

is provable in the classical sentential calculus; hence $\Gamma$ is not itself a matrix for the classical sentential calculus. From these two facts it is easily seen to follow that there cannot be a set of axioms for the classical sentential calculus, where each axiom contains only two variables.

The result obtained in the preceding paragraph can obviously be strengthened: If $A$ is a set of tautologies, which includes the tautology

$$[p \vee (q \vee r)] \rightarrow [(p \vee q) \vee r],$$

then $A$ cannot be derived from a set of axioms, each of which involves only two variables.[3] From this it is seen to follow that every set of axioms for the Heyting calculus, or for any one of the Łukasiewicz-Tarski calculi, must contain at least one axiom which involves three or more variables. A similar result is easily obtained for the Lewis systems S1, $\cdots$, S5.

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[3] It can be shown in a similar way that the tautology $p \rightarrow [q \rightarrow (r \rightarrow p)]$ is not derivable from tautologies each of which involves only two variables. This result was originally discovered by M. Wajsberg, and was stated without proof by J. Łukasiewicz and A. Tarski in Satz 15 of *Untersuchungen über den Aussagenkalkül*, Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, Classe III vol. 23 (1930) pp. 1–21.