BOOK REVIEWS

The theory of Lie groups, I. By Claude Chevalley. (Princeton Mathematical Series, no. 8.) Princeton University Press, 1946. 9+217 pp. $3.00.

In this masterpiece of concise exposition, the concept of Lie group is put together with all the craftsmanship of an expert. The finished product is a fascinating thing to contemplate, equipped as it is with three inter-related structures—algebraic, topological and analytic. It has no loose ends, no doubtful regions: it can be explored freely without the usual necessity of having to stay within a safe distance of the identity.

The reviewer was particularly struck that so much has been accomplished in a volume of such modest size. One of the things which makes this possible is the effectiveness of the definitions—invariably they emphasize the property which can be most quickly put to work and which is most suitable to the logic of the situation. This insistence on calling things by their right names can be disconcerting at times since it tends to ignore intuitive meanings. A tangent vector at the point p on an analytic manifold, for example, is defined as a certain type of mapping of the family of functions which are analytic at p. But in the end, this book should be easier for most readers, and far more satisfying, than any exposition of the same material proceeding on a less exact level; it should be a rugged base for volumes II, III, $\cdots$ which are to follow.

The book begins with a brief account of the orthogonal, unitary and symplectic groups of matrices and the generation of these groups by their infinitesimal elements. Then comes the important chapter on topological groups. This includes a remarkably sophisticated treatment of the covering concept. A covering of the space $\mathfrak{V}$ is a pair $(\tilde{\mathfrak{V}}, f)$ where f is a mapping $\tilde{\mathfrak{V}} \rightarrow \mathfrak{V}$ satisfying certain conditions of evenness. The coverings of topological groups are themselves converted into topological groups in a natural manner. A connected, locally connected space is called simply connected if it has no coverings other than itself (with f the identity mapping). If $\mathfrak{V}$ admits a covering $(\tilde{\mathfrak{V}}, f)$ in which $\tilde{\mathfrak{V}}$ is simply connected, it admits only one and therefore the group of automorphisms of $\tilde{\mathfrak{V}}$ which preserve f is determined only by $\mathfrak{V}$. This, by definition, is the Poincaré group of $\mathfrak{V}$. An existence theorem shows that a connected, locally connected space in which every point has a simply connected neighborhood has a simply connected covering. The author formulates and proves for simply connected spaces a "principle of monodromy" which he regards as the basic theorem for such spaces. A neat application yields at once the theorem that every local isomorphism of a simply connected group can be extended to a full isomorphism. As one might expect in this lofty treatment, it requires a certain amount of argument to show that a linear interval, for example, is simply connected. On the other hand, it is interesting to follow the development of these ideas without encountering anything as banal as a path or a homotopy. (It must be admitted, however, that paths are still very useful in finding one's way about in an unfamiliar space.)

Analyticity now enters the picture. It is perfectly possible to define an analytic group as a topological group such that some neighborhood of the identity can be covered with an n-dimensional coordinate system in terms of which the functions of group composition are analytic. This coordinate system, and those which are analytically related to it, can be transferred to all parts of the group by group translations and in this way the group becomes an analytic manifold in the sense of Whitney. The author prefers, for reasons of completeness and artistry, to develop the concept of analytic manifold *a priori* without the fortuitous aid of group properties. The definition is based on local families of analytic functions rather than local coordinates. An analytic group is now defined as a topological group whose underlying space is identified with an analytic manifold in such a way that the mapping $\mathfrak{V}\times\mathfrak{V}\rightarrow\mathfrak{V}$ given by $(\sigma, \eta)\rightarrow\sigma\eta^{-1}$ is everywhere analytic. A Lie group is a locally connected topological group $\mathfrak{G}$ in which the connected component which contains the identity is the underlying topological group of an analytic group. This analytic group is uniquely determined by the algebraic-topological structure of $\mathfrak{G}$, a remarkable fact which the author asserts does not hold when complex coordinate systems are allowed.

The chapter on analytic manifolds contains a careful treatment of the integral varieties of involutorial families of vector fields—that is, vector fields which are closed under the "bracket operation." In the case of an analytic group $\mathfrak{G}$, the vector fields which are important are those which are left-invariant under $\mathfrak{G}$; they form an involutorial system which is called the Lie algebra of $\mathfrak{G}$. The fundamental theorems of Lie show how a given group is related to its Lie algebra. This material is here developed and expanded in a manner wholly unhampered by tradition.

There follows a chapter on the Cartan calculus of differential forms. In an analytic group, the forms which are of especial interest are those which are left-invariant. In particular, the left-invariant

Pfaffian forms—called the forms of Maurer-Cartan—form a system which is dual to the left-invariant vector fields. The equations of Maurer-Cartan involve these forms and are a sort of dual to the equations of the so-called first fundamental theorem of Lie. This chapter on forms closes with the theory of invariant integration.

The last chapter deals with the representation theory for compact Lie groups and seems to the reviewer to be the most interesting from the scientific point of view. Let $\mathfrak{G}$ be a compact Lie group and let $\mathfrak{R}=\{P\mathfrak{V}\}$ be the totality of representations P of $\mathfrak{G}$ over the field C of complex numbers. $\mathfrak{R}$ has an algebraic structure of its own since it admits the operations $P_1\dot{+}P_2$ (summation), $P_1\times P_2$ (Kronecker multiplication), $\gamma P\gamma^{-1}$ (equivalence), $\overline{P}$ (passage to the complex conjugate). The coefficients of any representation are continuous complex-valued functions over $\mathfrak{G}$ and the totality of finite linear combinations of all coefficients of all representations is a ring $\mathfrak{o}$, called the representative ring of $\mathfrak{G}$. The two main objectives are (1) to show how $\mathfrak{G}$ can be reconstructed from $\mathfrak{R}$ and (2) to construct with the aid of $\mathfrak{o}$ an algebraic group in which $\mathfrak{G}$ can be isomorphically imbedded. We describe briefly the steps by which the author arrives at these goals simultaneously. A representation of $\mathfrak{R}$ is a function which associates to each P a matrix $\zeta(P)$ of the same size as P and this in such a way that sums, Kronecker products and equivalences are preserved. Matrix multiplication converts the totality of representations of $\mathfrak{R}$ into a group. Now every ring-homomorphism ω of $\mathfrak{o}$ into C defines a representation $\zeta(P)$ of $\mathfrak{R}$ (by assigning a complex number to each coefficient of each P, hence a matrix to each P) and conversely, every representation of $\mathfrak{R}$ comes from a homomorphism ω. Thus the totality $\{\omega\}$ is in one-one correspondence with the representations of $\mathfrak{R}$. This totality $\{\omega\}$ of homomorphisms is called the algebraic manifold $M(\mathfrak{G})$ associated with $\mathfrak{G}$. This may puzzle those who are not familiar with current notions in algebraic geometry. But the reason is simple. There exists at least one representation P of $\mathfrak{G}$ such that its d^2 coefficients $f_{ij}(\sigma)$ generate $\mathfrak{o}$. The f's will in general satisfy a number of algebraic equations and the complex numbers to which they correspond under a homomorphism ω must of course satisfy the same equations. Hence these numbers define a point on an algebraic variety in d^2-dimensional space and this variety is a "model" for $M(\mathfrak{G})$.—The correspondence which exists between $M(\mathfrak{G})$ and the representations of $\mathfrak{R}$ turns $M(\mathfrak{G})$ into a group, "the associated algebraic group" of $\mathfrak{G}$. Consider now the special homomorphisms ω in $M(\mathfrak{G})$ which preserve the passage to conjugate imaginaries. It is shown that there exists a natural isomorphism between this subgroup of $M(\mathfrak{G})$ and the original group $\mathfrak{G}$.

Hence $\mathfrak{G}$ can be thought of as imbedded isomorphically in its associated algebraic group $M(\mathfrak{G})$. The representations of $\mathfrak{R}$ to which these special ω's correspond are precisely those which preserve the passage to conjugate imaginaries (that is $\zeta(\bar{P})=\overline{\zeta(P)}$). This subgroup of the representations of $\mathfrak{R}$ is therefore in one-one correspondence with $\mathfrak{G}$. This is the duality theorem of Tannaka by which we regain $\mathfrak{G}$ from $\mathfrak{R}$. The usual theorems about reducibility, orthogonality and the approximation of continuous functions on $\mathfrak{G}$ are needed in the foregoing development; they are established with characteristic efficiency.

The book is dedicated, appropriately, to Elie Cartan and Hermann Weyl.

P. A. SMITH

Lectures on differential equations. By Solomon Lefschetz. (Annals of Mathematics Studies, no. 14.) Princeton University Press; London, Humphrey Milford, Oxford University Press, 1946. 8+210 pp. $3.00.

This book is a welcome addition to the literature of differential equations in the real domain, for in it one finds certain basic parts of the theory treated in a refreshingly modern manner. The principal topics considered include the fundamental existence and continuity theorems, critical points, periodic solutions, and the stability of solutions. Poincaré's geometrical theory of the qualitative properties of the general solution of an equation of the first order is developed at considerable length. One chapter is devoted particularly to systems of linear differential equations. It is the author's intention to furnish the necessary background for the modern work on the theory of nonlinear dynamical systems, and I believe that the judicious selection of the material, together with the thoroughness of the treatment, will cause the book to serve this purpose admirably. Some of the simpler physical applications are discussed briefly in the final chapter.

The consistent use of the terminology and properties of vector spaces, matrices, and matrix differential equations enables the author to handle quite general situations without any unduly complicated symbolism. The relevant parts of matrix theory are carefully explained in the first chapter; and the subsequent use of these notions results, for the most part, in a very perspicuous treatment of the subject matter. In a few places, mostly in the fourth and fifth chapters, the highly condensed notation, combined perhaps with some typographical errors, makes for rather difficult reading. (The book is lithoprinted from a typewritten manuscript and, as is usual in such cases, there are a goodly number of typographical errors. Most of these, however, are quite trivial, and will cause no difficulty.)