NOTE ON AFFINELY CONNECTED MANIFOLDS

SHIING-SHEN CHERN

The aim of this note is to prove some statements concerning the differential geometry in the large of an affinely connected manifold.

Let M be an orientable differentiable manifold of dimension n and class two. We say that an affine connection is defined in M, if a set of quantities[1] Γ^k_{ij} is defined in each allowable coordinate system x^i such that under change of the allowable coordinate system they are transformed according to the following law:

$$\bar{\Gamma}^r_{pq} = \frac{\partial^2 x^k}{\partial \bar{x}^p \partial \bar{x}^q} \frac{\partial \bar{x}^r}{\partial x^k} + \Gamma^k_{ij} \frac{\partial x^i}{\partial \bar{x}^p} \frac{\partial x^j}{\partial \bar{x}^q} \frac{\partial \bar{x}^r}{\partial x^k}. \tag{1}$$

The connection may be symmetric or asymmetric.

It is well known that from Γ^k_{ij} the covariant derivative of a contravariant vector X^i can be defined as follows:

$$X^i_{,k} = \frac{\partial X^i}{\partial x^k} + X^l \Gamma^i_{lk}. \tag{2}$$

We also recall that the affine curvature tensor is given by

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} - \Gamma^i_{ml}\Gamma^m_{jk} + \Gamma^i_{mk}\Gamma^m_{jl}. \tag{3}$$

We put

$$R_{kl} = R^i_{ikl} \tag{4}$$

and introduce the exterior differential form

$$P = R_{kl} dx^k dx^l. \tag{5}$$

Then the main theorem of this note can be stated as follows:

THEOREM. *The integral of P over any two-dimensional cycle is equal to zero.*

To prove this theorem we consider n linearly independent contravariant vectors $X^i_{(1)}, \cdots, X^i_{(n)}$ and their determinant

Received by the editors February 8, 1947.

[1] All indices in this paper run from 1 to n and we agree as usual that repeated indices mean summation.

$$\Delta = \epsilon_{i_1 \cdots i_n} X^{i_1}_{(1)} \cdots X^{i_n}_{(n)}, \tag{6}$$

where $\epsilon_{i_1 \cdots i_n}$ is the Kronecker index. When the vectors are parallelly transported, we have

$$-\frac{d\Delta}{\Delta} = \Gamma^{i}_{ik} dx^{k}. \tag{7}$$

The differential form $-d\Delta/\Delta$ we shall denote for simplicity by ϕ. It now follows immediately from (3) that its exterior derivative is

$$d\phi = 2P. \tag{8}$$

It is sufficient for the proof of this theorem to assume that the domain of integration is a simplicial two-dimensional cycle in a sufficiently fine simplicial decomposition of M such that each simplex lies in one coordinate neighborhood of M, as every cycle over which the integral of P is defined can be approximated by a simplicial cycle with the above property. On such a simplicial cycle we shall define a continuous field of n independent contravariant vectors at each point. According to a standard procedure in the theory of fibre bundles,[2] the sets of vectors are at first defined at the vertices of the cycle. They can be extended over the one-dimensional simplexes, because M is orientable. To extend the field over the two-dimensional simplexes and hence over the whole cycle, we notice that the sets of n independent contravariant vectors at a point whose determinant is positive (or negative) form a topological space which is the group space of the group of linear transformations in n variables with positive (or negative) determinant and is hence simply-connected.[3] As the sets of vectors defined on the boundary of a simplex give rise to a mapping of the boundary into the group space in question, the simply-connectedness of this group space implies the possibility of extension over the two-dimensional simplexes. The integral of P over the cycle can now be interpreted as the integral of P over this field of independent contravariant vectors in the space of all sets of n independent contravariant vectors of the manifold M. From (8) it follows that the integral is zero and the theorem is proved.

With the terminology of the cohomology theory the theorem can be stated as follows: *The cohomology class to which P belongs is the zero class.*

We shall state the following corollaries of our theorem:

[2] Cf., for instance, N. Steenrod, *Topological methods for the construction of tensor functions*, Ann. of Math. vol. 43 (1942) pp. 116–131.

[3] A proof is given later in this paper.

COROLLARY 1. *Let M be a compact orientable affinely-connected manifold of class two and even dimension n. Then*

$$\int_M P^{n/2} = 0, \tag{9}$$

or

$$\int_M \epsilon^{i_1 \cdots i_n} R_{i_1 i_2} R_{i_3 i_4} \cdots R_{i_{n-1} i_n} dx^1 \cdots dx^n = 0. \tag{9a}$$

The integrand of the integral (9a):

$$I = \epsilon^{i_1 \cdots i_n} R_{i_1 i_2} R_{i_3 i_4} \cdots R_{i_{n-1} i_n}$$

is a tensor density of the affine connection.

COROLLARY 2. *Let M have the same meaning as in Corollary* 1. *Then I can not be always positive nor always negative in M.*

Remark. In terms of the notation of Elie Cartan, the affine connection can be written as the infinitesimal displacement of affine frames $p\mathfrak{e}_1 \cdots \mathfrak{e}_n$:

$$dp = \omega^i \mathfrak{e}_i, \qquad d\mathfrak{e}_i = \omega_i^j \mathfrak{e}_j, \tag{10}$$

with the equations of structure:

$$d\omega^i = \omega^j \omega_j^i + \Omega^i, \qquad d\omega_j^i = \omega_j^k \omega_k^i + \Omega_j^i. \tag{11}$$

Then we have

$$P = \Omega_i^i. \tag{12}$$

I shall conjecture that all the differential forms

$$P_\alpha = \Omega_{i_1}^{i_2} \Omega_{i_2}^{i_3} \cdots \Omega_{i_\alpha}^{i_1}, \qquad 0 \leqq \alpha \leqq n/4, \tag{13}$$

define cohomology classes which are topological invariants of M. I hope to return to this in a later paper.

For completeness I indicate here a proof of the well known fact that the space of all matrices (with a natural topology) (a_{ij}), $|a_{ij}| > 0$, is simply connected, as I am not able to give the exact reference. This space is a group manifold and the theorem in question follows from the following theorem which I quote from Chevalley in a slightly different version:[4]

[4] C. Chevalley, *Theory of Lie groups*, p. 59.

Let G be a connected Lie group and H a subgroup of G. If the homogeneous space G/H is simply connected, the fundamental group of G is isomorphic to a factor group of the fundamental group of H.

Using this theorem we shall give the proof by induction on the order n of the matrices. Let G be the group of all these matrices, H the subgroup of all matrices

$$\begin{pmatrix} 1 & a_{12} \cdots a_{1n} \\ 0 & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 & a_{n2} \cdots a_{nn} \end{pmatrix}$$

and K the subgroup of all matrices

$$\begin{pmatrix} 1 & 0 \quad \cdots 0 \\ 0 & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \quad \cdot \\ 0 & a_{n2} \cdots a_{nn} \end{pmatrix},$$

so that

$$K \subset H \subset G.$$

The induction hypothesis states that K is simply connected. It is easy to verify that both homogeneous spaces H/K (the space of right cosets) and G/H (the space of left cosets) are Euclidean spaces and hence are simply connected. It therefore follows from the above theorem that H and hence also G are simply connected.

INSTITUTE OF MATHEMATICS, ACADEMIA SINICA