ON THE CHARACTERISTIC EQUATIONS OF CERTAIN MATRICES

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In a paper to be published soon in the Annals of Mathematical Statistics, R. v. Mises obtains the following theorem on matrices from results in the theory of probability.

THEOREM. *Let $A=(a_{\kappa\lambda})$, $B=(b_{\kappa\lambda})$, and $C=(c_{\kappa\lambda})$ be square matrices of order n. If the elements of A and C satisfy the conditions*

$$r_\kappa = \sum_{\nu=1}^{n} a_{\kappa\nu} = 0 \qquad (\kappa = 1, 2, \cdots, n), \tag{1}$$

$$s_\lambda = \sum_{\nu=1}^{n} a_{\nu\lambda} = 0 \qquad (\lambda = 1, 2, \cdots, n), \tag{2}$$

$$c_{\kappa\lambda} = c_\kappa + c_\lambda \qquad (\kappa, \lambda = 1, 2, \cdots, n) \tag{3}$$

where $c_1, c_2, \cdots, c_n$ are arbitrary numbers, then the matrices AB and $A(B+C)$ have the same characteristic equation.

In the following a purely algebraic proof of this theorem will be given.

PROOF. We set

$$\sum_{\nu=1}^{n} a_{\kappa\nu} c_\nu = q_\kappa \qquad (\kappa = 1, 2, \cdots, n).$$

Then we have by (1) and (3)

$$AC = \begin{pmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdot & \cdot \quad \cdot \quad \cdot \\ a_{n1} & a_{n2} \cdots a_{nn} \end{pmatrix} \begin{pmatrix} c_1+c_1 & c_1+c_2 \cdots c_1+c_n \\ c_2+c_1 & c_2+c_2 \cdots c_2+c_n \\ \cdot & \cdot \quad \cdot \quad \cdot \\ c_n+c_1 & c_n+c_2 \cdots c_n+c_n \end{pmatrix}$$
$$= \begin{pmatrix} q_1+c_1r_1 & q_1+c_2r_1 \cdots q_1+c_nr_1 \\ q_2+c_1r_2 & q_2+c_2r_2 \cdots q_2+c_nr_2 \\ \cdot & \cdot \quad \cdot \quad \cdot \\ q_n+c_1r_n & q_n+c_2r_n \cdots q_n+c_nr_n \end{pmatrix} = \begin{pmatrix} q_1 & q_1 \cdots q_1 \\ q_2 & q_2 \cdots q_2 \\ \cdot & \cdot \quad \cdot \\ q_n & q_n \cdots q_n \end{pmatrix}. \tag{4}$$

Let P be the triangular matrix

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$$P = \begin{pmatrix} 1 & 0 & 0 & 0\cdots 0 \\ 1 & 1 & 0 & 0\cdots 0 \\ 1 & 1 & 1 & 0\cdots 0 \\ \cdot & \cdot & \cdot & \cdot\ \cdot\ \cdot \\ 1 & 1 & 1 & 1\cdots 1 \end{pmatrix}; \quad P^{-1} = \begin{pmatrix} 1 & 0 & 0\cdots 0 \\ -1 & 1 & 0\cdots 0 \\ 0 & -1 & 1\cdots 0 \\ \cdot & \cdot & \cdot\ \cdot\ \cdot \\ 0 & 0\cdots 0 & -1\quad 1 \end{pmatrix}.$$

We have by (4)

$$\begin{aligned} PAC &= \begin{pmatrix} 1 & 0\cdots & 0 \\ 1 & 1\cdots & 0 \\ \cdot & \cdot & \cdot \\ 1 & 1\cdots 1 & 0 \\ 1 & 1\cdots 1 & 1 \end{pmatrix} \begin{pmatrix} q_1 & q_1 & \cdots & q_1 \\ q_2 & q_2 & \cdots & q_2 \\ \cdot & \cdot & \cdot & \cdot \\ q_{n-1} & q_{n-1} & \cdots & q_{n-1} \\ q_n & q_n & \cdots & q_n \end{pmatrix} \\ &= \begin{pmatrix} q_1 & q_1 & \cdots & q_1 \\ q_1+q_2 & q_1+q_2 & \cdots & q_1+q_2 \\ \cdot & \cdot & \cdot & \cdot \\ q_1+q_2+\cdots+q_{n-1} & q_1+q_2+\cdots+q_{n-1} & \cdots & q_1+q_2+\cdots+q_{n-1} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \end{aligned} \tag{5}$$

since by (2)

$$\sum_{\nu=1}^{n} q_\nu = \sum_{\nu=1}^{n}\sum_{\lambda=1}^{n} a_{\nu\lambda}c_\lambda = \sum_{\lambda=1}^{n} c_\lambda \sum_{\nu=1}^{n} a_{\nu\lambda} = \sum_{\lambda=1}^{n} c_\lambda s_\lambda = 0.$$

Hence

$$\begin{aligned} PACP^{-1} &= \begin{pmatrix} q_1 & q_1 & \cdots & q_1 \\ q_1+q_2 & q_1+q_2 & \cdots & q_1+q_2 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & 0\cdots & 0 & 0 \\ -1 & 1\cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0\cdots & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0\cdots 0 & q_1 \\ 0 & 0\cdots 0 & q_1+q_2 \\ \cdot & \cdot & \cdot \\ 0 & 0\cdots 0 & q_1+q_2+\cdots+q_{n-1} \\ 0 & 0\cdots 0 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

On the other hand, it follows from (2), similarly as in (5), that PA, and therefore also PAB and $PABP^{-1}$, are matrices in which all the elements of the last row are equal to 0. Hence $PABP^{-1}$ has the form

$$PABP^{-1} = \left(\begin{array}{ccc|c} & & & t_1 \\ & & & t_2 \\ & D_{nn} & & \vdots \\ & & & t_{n-1} \\ \hline 0 & 0 \cdots 0 & & 0 \end{array}\right) \tag{7}$$

where D_{nn} is a square matrix of order $n-1$ and $t_1, t_2, \cdots, t_{n-1}$ are certain elements. It follows from (7) and (6) that

$$PA(B+C)P^{-1} = \left(\begin{array}{ccc|c} & & & t_1 + q_1 \\ & & & t_2 + q_1 + q_2 \\ & D_{nn} & & \vdots \\ & & & t_{n-1} + q_1 + q_2 + \cdots + q_{n-1} \\ \hline 0 & 0 \cdots 0 & & 0 \end{array}\right). \tag{8}$$

If we denote the characteristic polynomial of the matrix D_{nn} by $f(x)$, then it follows from (7) and (8) that $PABP^{-1}$ and $PA(B+C)P^{-1}$ both have the characteristic equation

$$xf(x) = 0. \tag{9}$$

Since similar matrices have the same characteristic equation, (9) is also the characteristic equation of AB and $A(B+C)$.

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