A SUM CONNECTED WITH THE PARTITION FUNCTION

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1. **Introduction.** In the formula for the number $p(n)$ of unrestricted partitions of an integer n there appears the sum [1][1]

$$A_k(n) = \sum_{h \bmod k}{}' \omega_{h,k} \exp(-2\pi i h n/k), \tag{1}$$

where the dash ′ beside the summation symbol indicates here and in the sequel that the letter of summation runs only through a reduced residue system with respect to the modulus. The symbol $\omega_{h,k}$ denotes certain $24k$th roots of unity given by

$$\omega_{h,k} = \exp(\pi i s(h, k)),$$

where $s(h, k)$ is a Dedekind sum [2] defined by

$$s(h, k) = \sum_{\mu=1}^{k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{\mu h}{k}\right)\right).$$

The symbol $((x))$, in turn, is defined as follows:

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{for } x \text{ not an integer,} \\ 0 & \text{for } x \text{ an integer,} \end{cases}$$

where $[x]$ denotes, as usual, the greatest integer not exceeding x. D. H. Lehmer [3] has investigated these sums on the basis of a different expression for the roots of unity involved. In the first place he factored the $A_k(n)$ according to the prime powers contained in k. Secondly, by reducing them to sums studied by H. D. Kloosterman [4] and H. Salié [5], he evaluated the $A_k(n)$ explicitly in the case in which k is a prime or a power of a prime. Both results together provide a method for calculating the $A_k(n)$. Alternate proofs of Lehmer's factorization theorems have been given in [2]. In the present note a new approach to the second of Lehmer's results is presented. The method given here is simpler than Lehmer's method, especially in the treatment of the case $k = 2^\lambda$.

2. **Some lemmas.** The proofs are based in part on three lemmas, which occur as Theorems 17, 18, 19 in [2].

Lemma 1. *Let $\theta = \theta(k)$ denote 1 for $3 \nmid k$ and 3 for $3 \mid k$ so that θk is*

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

either prime to 3 or divisible by 3^2. For $(h, k)=1$ we have

$$12hks(h, k) \equiv h^2 + 1 \pmod{\theta k}.$$

Moreover

$$12ks(h, k) \equiv 0 \pmod 3$$

if and only if $3 \nmid k$.

LEMMA 2. *For odd k we have $12ks(h, k) \equiv k+1-2(h|k) \pmod 8$, where $(h|k)$ denotes the Legendre-Jacobi symbol.*

LEMMA 3. *If k is equal to $2^\lambda j$, $\lambda \geqq 0$, and j and h are odd integers, then*

$$12hks(h, k) \equiv h^2 + k^2 + 3k + 1 + 2k(k \mid h) \pmod{2^{\lambda+3}}.$$

The following lemmas concerning generalized Kloosterman sums are also needed.

LEMMA 4. *Let $k=p^\alpha$, where p is an odd prime. Put*

$$S_k(n) = \sum_{h \bmod k}' (h \mid k) \exp (2\pi i(nh + \bar{h})/k),$$

where $\bar{h}$ is defined as any solution of the congruence $h\bar{h} \equiv 1 \pmod k$. Then

$$S_k(n) = \begin{cases} 0 & \text{if } n \text{ is a non-residue of } k \text{ prime to } k, \\ 2i^{((k-1)/2)^2}k^{1/2} \cos 4\pi m/k & \text{if } m^2 \equiv n \pmod k,\ n \text{ prime to } k, \\ 0 & \text{if } n \text{ is divisible by } p \text{ and } \alpha > 1, \\ i^{((k-1)/2)^2}k^{1/2} & \text{if } n \text{ is divisible by } p \text{ and } \alpha = 1. \end{cases}$$

LEMMA 5. *Let $k=p^\beta$, $\beta>1$, and $n\equiv 1 \pmod p$, where p denotes an odd prime congruent to $3 \pmod 4$. Then*

$$\sum_{h \bmod k}' (h \mid p^{\beta-1}) \exp (2\pi i(nh + \bar{h})/k) = 2i^{((k-1)/2)^2+1}(m \mid p) k^{1/2} \sin (4\pi m/k),$$

where m is an integer such that $m^2 \equiv n \pmod k$.

Lemmas 4 and 5 may be proved by employing a method due to the author [6]. The method carries over step by step with only slight modifications and it does not seem necessary to present it again here.

LEMMA 6. *If k is equal to 2^λ, $\lambda \geqq 9$, and n is an odd integer, then*

$$\sum_{h \bmod k}' (2k \mid h) \exp (2\pi in(h + \bar{h})/k) = 2(k \mid n)(2k)^{1/2} \cos (\pi/4 + (-1)^{(n-1)/2}4\pi n/k).$$

Lemma 6 may be established easily if use is made of Salié's [5] discussion (§3) of the corresponding sum without a quadratic character. In fact, the introduction of the character has no influence on the argument. It is therefore not necessary to reproduce the proof here.

3. **Proof of Lehmer's theorems.** In order to apply the lemmas of the preceding section to the evaluation of $A_k(n)$ for k a power of an odd prime p, we divide the discussion into two cases according as $p>3$ or $p=3$. For $p>3$ we employ the congruence

$$\begin{aligned} 6k[-((k-1)/2)^2 - 1 + (2h \mid k)] + 24^2ah + 24\cdot\overline{24}\bar{h} \\ \equiv 12ks(h, k) - 24hn \pmod{24k}, \end{aligned} \tag{2}$$

where a and $\bar{h}$ are defined by means of the congruences

$$24^2a \equiv 1 - 24n \pmod{k}, \qquad h\bar{h} \equiv 1 \pmod{k}. \tag{3}$$

Divisibility with respect to k and with respect to 3 in congruence (2) follows at once from Lemma 1. Divisibility with respect to 8 follows from Lemma 2 and the well known relation $(2 \mid k) = (-1)^{(k^2-1)/8}$. Hence, from (1) and (2), we get

$$\begin{aligned} A_k(n) &= \sum_{h \bmod k}{}' \exp\left[2\pi i(ks(h, k)/2 - hn)/k\right] \\ &= \sum_{h \bmod k}{}' \exp\left\{2\pi i[k(-((k-1)/2)^2 - 1 + (2h \mid k))/4\right. \\ &\qquad\qquad\qquad\qquad \left. + 24ah + \overline{24}\bar{h}]/k\right\} \\ &= (-i)^{((k-1)/2)^2}(2 \mid k) \sum_{h \bmod k}{}' (h \mid k) \exp\left[2\pi i(24ah + \overline{24}\bar{h})/k\right] \\ &= (-i)^{((k-1)/2)^2}(3 \mid k) \sum_{h \bmod k}{}' (h \mid k) \exp\left[2\pi i(ah + \bar{h})/k\right]. \end{aligned}$$

Applying Lemma 4, and using (3), we obtain the following theorem.

THEOREM 1. *If $k=p^\alpha$, $\alpha \geqq 1$, and $v=1-24n$, where p is a prime greater than 3, then*

$$A_k(n) = \begin{cases} 0 & \text{if } v \text{ is a non-residue}^2 \text{ of } k, \text{ prime to } k, \\ 2(3 \mid k)k^{1/2} \cos(4\pi m/k) & \text{if } v \equiv (24m)^2 \pmod{k}, \text{ prime to } k, \\ 0 & \text{if } v \equiv 0 \pmod{p} \text{ and } \alpha > 1, \\ (3 \mid k)k^{1/2} & \text{if } v \equiv 0 \pmod{p} \text{ and } \alpha = 1. \end{cases}$$

We turn next to the case which arises when $k=p^\alpha$ and $p=3$. This

[2] This condition should not be confused with $(v \mid k) = -1$. We mean that no solution exists of the congruence $x^2 \equiv v \pmod{k}$.

time we use the congruence

$$(4)\quad \begin{aligned} 6k[-((k-1)/2)^2 - 1 + (2h\mid k)] + 8^2ah + 8\cdot\bar{8}\bar{h} \\ \equiv 12ks(h,k) - 24hn \pmod{24k}, \end{aligned}$$

where a and $\bar{h}$ are defined by the congruences

$$(5)\qquad 8^2a \equiv 1 - 24n \pmod{3k}, \qquad h\bar{h} \equiv 1 \pmod{3k}.$$

Divisibility with respect to $3k$ in congruence (4) follows immediately from Lemma 1. Divisibility with respect to 8 follows from Lemma 2. Hence, by (1) and (4), we have

$$\begin{aligned} A_k(n) &= \sum_{h \bmod k}{}' \exp\,[2\pi i(3ks(h,k)/2 - 3hn)/3k] \\ &= \sum_{h \bmod k}{}' \exp\,\{2\pi i[3k(-((k-1)/2)^2 - 1 + (2h\mid k))/4 \\ &\qquad\qquad + 8ah + \bar{8}\bar{h}]/3k\} \\ &= (-i)^{((k-1)/2)^2}(2\mid k) \sum_{h \bmod k}{}' (h\mid k) \exp\,[2\pi i(8ah + \bar{8}\bar{h})/3k]. \end{aligned}$$

As h runs through a reduced residue system mod k, so does $h+k$. Replace $\bar{h}$ by $\bar{h}-k\bar{h}^2$, and observe that (5) implies that

$$8a(h+k) + \bar{8}(\bar{h} - k\bar{h}^2) \equiv 8ah + \bar{8}\bar{h} \pmod{3k}.$$

If h now runs over a reduced residue system mod $3k$, instead of mod k, we obtain $3A_k(n)$ instead of $A_k(n)$. Therefore,

$$\begin{aligned} A_k(n) &= \frac{1}{3}(-i)^{((k-1)/2)^2}(2\mid k) \sum_{h \bmod 3k}{}' (h\mid k) \exp\,[2\pi i(8ah + \bar{8}\bar{h})/3k] \\ &= \frac{1}{3}(-i)^{((k-1)/2)^2} \sum_{h \bmod 3k}{}' (h\mid k) \exp\,[2\pi i(ah + \bar{h})/3k]. \end{aligned}$$

Since $a \equiv 1 \pmod 3$, we may apply Lemma 5 with β replaced by $\alpha+1$. Thus we get the following theorem.

THEOREM 2. *If* $k = 3^\beta$, *then*

$$A_k(n) = 2(-1)^{\beta+1}(m\mid 3)(k/3)^{1/2}\sin\,(4\pi m/3k),$$

where m *is an integer such that* $(8m)^2 \equiv 1 - 24n \pmod{3k}$.

Finally, we consider the case $k = 2^\lambda$. For this purpose we use the congruence

$$(6)\quad 3\bar{h}k - 6\bar{h}k(k\mid h) + 3^2ah + 3\cdot\bar{3}\bar{h} \equiv 12ks(h,k) - 24hn \pmod{24k},$$

where a and $\bar{h}$ are defined by the congruences

$$(7) \qquad 3^2a \equiv 1 - 24n \pmod{8k}, \qquad h\bar{h} \equiv 1 \pmod{8k},$$

and where we have assumed that $\lambda \geqq 3$.

Divisibility with respect to 3 in congruence (6) follows from the second part of Lemma 1. Divisibility with respect to $8k$ follows from Lemma 3. From (1) and (6) we now obtain

$$\begin{aligned} A_k(n) &= \sum_{h \bmod k}' \exp\left[2\pi i(4ks(h, k) - 8hn)/8k\right] \\ &= \sum_{h \bmod k}' \exp\left[2\pi i(\bar{h}k - 2\bar{h}k(k \mid h) + 3ah + 3\bar{h})/8k\right] \\ &= \frac{1}{8} \sum_{h \bmod 8k}' \exp\left[2\pi i(h - 2h(k \mid h))/8\right] \exp\left[2\pi i(3ah + 3\bar{h})/8k\right]. \end{aligned}$$

It is easy to verify that

$$\exp\left[2\pi i(h - 2h(k \mid h))/8\right] = (-1)^{\lambda+1}(k \mid 3h) \exp\left[2\pi i(3h)/8\right].$$

Therefore,

$$(8) \quad A_k(n) = \frac{1}{8}(-1)^{\lambda+1} \sum_{h \bmod 8k}' (k \mid 3h) \exp\left[2\pi i(3ah + 3\bar{h}(1 + k))/8k\right].$$

Now, for $\lambda \geqq 5$, we have $(1+k/2)^2 \equiv 1+k \pmod{8k}$. Furthermore, it follows from (7) that a is a quadratic residue of $2^{\lambda+3}$ since $3^2a \equiv 1 \pmod 8$. Let $m^2 \equiv a \pmod{8k}$. Replacing h by $\bar{3}\bar{m}(1+k/2)h$, and applying Lemma 6, we obtain for $\lambda \geqq 6$,

$$\begin{aligned} A_k(n) &= \frac{1}{8}(-1)^{\lambda+1}(k \mid m) \sum_{h \bmod 8k}' (k \mid h) \exp\left[2\pi i m(1 + k/2)(h + \bar{h})/8k\right] \\ &= (-1)^{\lambda+1}(2 \mid m) k^{1/2} \cos\left\{\pi/4 + (-1)^{(m-1)/2} 4\pi[m(1 + k/2)]/8k\right). \end{aligned}$$

The last equation may be simplified by employing the relation $(2 | m) = (-1)^{(m^2-1)/8}$. We obtain thus the following theorem.

THEOREM 3. *If* $k = 2^\lambda$, $\lambda \geqq 0$, *then*

$$A_k(n) = (-1)^\lambda(-1 \mid m) k^{1/2} \sin(4\pi m/8k),$$

where m is an integer such that $(3m)^2 \equiv 1 - 24n \pmod{8k}$.

Actually we have established Theorem 3 only for $\lambda \geqq 6$. The verification of this theorem for $\lambda < 6$ is left as an exercise for the reader. For $0 \leqq \lambda \leqq 3$, use the definition of $A_k(n)$ given by (1). For $\lambda = 4, 5$ use the formula for $A_k(n)$ given by (8).

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