of Lemma 3 to obtain suitable θ's for groups of the form $Z_1 \times Z_2 \times Z_3$ where Z_i are cyclic of order 2^{n_i}. However, it should be noted that if $G \cong G_1 \times G_2$, a one-to-one mapping θ of G upon G may be defined by

$$\theta[(x, y)] = [\theta_1(x), \theta_2(y)]$$

where θ_1 and θ_2 are one-to-one mappings of G_1 upon G_1 and G_2 upon G_2 respectively. Moreover θ satisfies the relationship $O(\eta) \geqq O(\eta_1) \cdot O(\eta_2)$. Thus if $O(\eta_1) = n(G_1)$, $O(\eta_2) = n(G_2)$ we would have $O(\eta) = n(G_1 \times G_2)$ and θ is represented explicitly in terms of θ_1 and θ_2.

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ON RINGS WHOSE ASSOCIATED LIE RINGS ARE NILPOTENT

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1. Introduction. With any ring R we may associate a Lie ring $(R)_l$ by combining the elements of R under addition and commutation, where the commutator $x \circ y$ of two elements $x, y \in R$ is defined by

$$x \circ y = xy - yx.$$

We call $(R)_l$ the Lie ring associated with R, and denote it by $\mathfrak{R}$. The question of how far the properties of $\mathfrak{R}$ determine those of R is of considerable interest, and has been studied extensively for the case when R is an algebra, but little is known of the situation in general. In an earlier paper the author investigated the effect of the nilpotency of $\mathfrak{R}$ upon the structure of R if R contains a nilpotent ideal N such that R/N is commutative.[1] In the present note we prove that, for an arbitrary ring R, the nilpotency of $\mathfrak{R}$ implies that the commutators of R of the form $x \circ y$ generate a nil-ideal, while the commutators of R of the form $(x \circ y) \circ z$ generate a nilpotent ideal (cf. §3). If R is finitely generated, and $\mathfrak{R}$ is nilpotent then the ideal generated by the commutators $x \circ y$ is also nilpotent (cf. §4).

2. A lemma on L-nilpotent rings. We recall that the Lie ring $\mathfrak{R}$ is said to be nilpotent of class γ if we have

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[1] *Central chains of ideals in an associative ring*, Duke Math. J. vol. 9 (1942) pp. 341–355, Theorem 6.5.

$$\mathfrak{R} = \mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \cdots \supset \mathfrak{R}_\gamma \supset \mathfrak{R}_{\gamma+1} = 0 \tag{1}$$

where $\mathfrak{R}_k = [\mathfrak{R}_{k-1}, \mathfrak{R}]$ is the Lie ideal of $\mathfrak{R}$ generated by all elements of the form $x \circ y$ with $x \in \mathfrak{R}$ and $y \in \mathfrak{R}_{k-1}$. If R is a ring whose associated Lie ring is nilpotent of class γ then we shall say that R is *L-nilpotent of class* γ. It is well known that the lower central chain (1) has the property $[\mathfrak{R}_\lambda, \mathfrak{R}_\mu] \subseteq \mathfrak{R}_{\lambda+\mu}$ and hence in particular

$$[\mathfrak{R}_\lambda, \mathfrak{R}_\lambda] = 0 \qquad \text{if } 2\lambda > \gamma. \tag{2}$$

We prove the following lemma.

LEMMA 1. *Let R be an L-nilpotent ring of class γ. If $c \in \mathfrak{R}_{\gamma-1}$ and if x, y are arbitrary elements of R then*

$$(c \circ x)(c \circ y) = 0,$$

and in particular

$$(c \circ x)^2 = 0.$$

If $c_1, c_2 \in \mathfrak{R}_{\gamma-1}$ and $c_1 \circ c_2 = 0$ then for arbitrary $x, y \in R$

$$(c_1 \circ x)(c_2 \circ y) = 0.$$

PROOF. Consider the identity

$$(\overline{a \circ by} \circ x) = (\overline{a \circ b} \circ x)y + (a \circ b)(y \circ x) + b(\overline{a \circ y} \circ x) + (b \circ x)(a \circ y).$$

Setting $a = b = c$ we have, since $[\mathfrak{R}_{\gamma-1}, \mathfrak{R}, \mathfrak{R}] = 0$,

$$0 = (c \circ x)(c \circ y)$$

and, if $x = y$,

$$0 = (c \circ x)^2,$$

while if $a = c_2$, $b = c_1$ and $c_1 \circ c_2 = 0$

$$0 = (c_1 \circ x)(c_2 \circ y),$$

which proves the lemma.

3. **Ideals generated by the lower central chain of $\mathfrak{R}$.** In what follows, R will be an L-nilpotent ring, and we denote the lower central chain of $\mathfrak{R}$ as in (1). Let R_k, $k = 1, 2, \cdots, \gamma$, be the subring of R generated by the elements of $\mathfrak{R}_k$, and let $\overline{R}_k$ be the ideal of R generated by R_k. It is known[2] that every element of $\overline{R}_k$ may be written in the form $u_k + v_k$, where $u_k \in R_k$ and $v_k \in RR_k$, and since R_γ is in the centre of R, $\overline{R}_\gamma$ is a nilpotent or nil-ring whenever R_γ is.

[2] Ibid. Lemma 5.3.

Let $R^* = R/\overline{R}_\gamma$; then the natural homomorphism of R upon R^* induces a homomorphism of $\mathfrak{R}$ upon $\mathfrak{R}^*$, where $\mathfrak{R}^*$ is the Lie ring associated with R^*, such that $\overline{R}_k \to \overline{R}_k^*$. Hence in particular $\mathfrak{R}_\gamma^* = 0$ and R^* is an L-nilpotent ring of class not greater than $\gamma - 1$.

Our principal theorem is the following:

THEOREM 1. *If R is an L-nilpotent ring, then the commutators of R generate a nil-ideal of R, that is, $\overline{R}_2$ is a nil-ideal. The elements of R of the form $(x \circ y) \circ z$ generate a nilpotent ideal of R, that is, $\overline{R}_3$ is nilpotent.*

PROOF. Consider first R_γ: every element of R_γ can be written as a finite sum of finite products of elements of $\mathfrak{R}_\gamma$ and since $\mathfrak{R}_\gamma = [\mathfrak{R}_{\gamma-1}, \mathfrak{R}]$, every element of $\mathfrak{R}_\gamma$ can be written as a finite sum of elements of the form $c \circ x$, where $c \in \mathfrak{R}_{\gamma-1}$ and $x \in R$. Hence every element of R_γ is a sum of products of elements of the form $c \circ x$. Now by Lemma 1 the square of every element of the form $c \circ x$ is zero, and these elements are all in the centre of R. Hence if

$$y = p_1 + p_2 + \cdots + p_n$$

is an element of R_γ, where the p_k are products of elements of the form $c \circ x$, we have $p_k^2 = 0$ and therefore, since these products p_k are all in the centre of R, we have $y^{n+1} = 0$, which proves that R_γ, and hence $\overline{R}_\gamma$, is a nil-ring. Now if $\gamma > 2$ we have, from (2)

$$[\mathfrak{R}_{\gamma-1}, \mathfrak{R}_{\gamma-1}] = 0$$

and hence $c_1 \circ c_2 = 0$ for all $c_1, c_2 \in \mathfrak{R}_{\gamma-1}$. From Lemma 1 it follows that $p_i p_j = 0$ in the representation of y above, and hence

$$\overline{R}_\gamma^2 = 0, \qquad \gamma > 2.$$

The proof of the theorem now proceeds easily by induction upon γ, since by the above it is true when $\gamma = 2$, that is whenever $\overline{R}_2 = \overline{R}_\gamma$, $\overline{R}_3 = 0$. We suppose, therefore, that the theorem holds for rings of class less than γ, and hence in particular for $R^* = R/\overline{R}_\gamma$. Then if $c \in \overline{R}_2$ and $c \to c^*$ in the homomorphism of R upon R^* we have

$$c^{*\sigma'} = 0, \qquad \sigma' \text{ some integer,}$$

by our induction, and hence

$$c^{\sigma'} \in \overline{R}_\gamma \qquad \text{for all } c \in \overline{R}_2.$$

Since $\overline{R}_\gamma^2 = 0$ whenever $\gamma > 2$ we have

$$c^\sigma = 0, \qquad \text{where } \sigma = 2\sigma',$$

and it follows that $\overline{R}_2$ is a nil-ring. Further, since $\overline{R}_3^*$ is nilpotent by our induction,

$$\overline{R}_3^{*\tau'} = 0 \qquad \text{for some integer } \tau'$$

and hence

$$\overline{R}_3^{\tau'} \subseteq \overline{R}_\gamma$$

and therefore

$$\overline{R}_3^{\tau} = 0, \qquad \text{where } \tau = 2\tau',$$

which proves that $\overline{R}_3$ is nilpotent, as required.

4. **Finitely generated L-nilpotent rings.** If R satisfies the maximal or minimal condition for one-sided ideals, so does $\overline{R}_2$ and hence $\overline{R}_2$ must be nilpotent.[8] We prove the following stronger result:

THEOREM 2. *If R is a finitely generated L-nilpotent ring, then the commutators of R generate a nilpotent ideal, that is, $\overline{R}_2$ is nilpotent.*

PROOF. If R is finitely generated, say by $x_1, x_2, \cdots, x_d$, then every element x of R can be written in the form $x = p_1 + p_2 + \cdots + p_n$ where the p_k are products of the $x_1, \cdots, x_d$ in some order. It is clearly sufficient to consider the case $\gamma = 2$, since if we show in general that $\overline{R}_2/\overline{R}_3$ is nilpotent, it will follow from Theorem 1 that $\overline{R}_2$ has this property. Because of the identity

$$(ab) \circ c = a(b \circ c) + (a \circ c)b$$

every element of $\overline{R}_2$ can be written as a sum of products of the form

$$\pi_r = a(x_{i_1} \circ x_{j_1})(x_{i_2} \circ x_{j_2}) \cdots (x_{i_r} \circ x_{j_r}), \qquad a \in R.$$

Now there are at most $d(d-1)/2$ nonzero commutators of the type $x_i \circ x_j$, and since by Lemma 1 we have

$$(x_i \circ x_j)(x_i \circ x_k) = 0$$

it follows that if the number of factors in any product π_r is greater than $d(d-1)/2$ this product vanishes. Hence

$$\overline{R}_2^{\tau} = 0, \qquad \tau = d(d-1)/2 + 1$$

and the theorem is established.

[8] Cf. C. Hopkins, *Nilrings with minimum condition for admissible left ideals*, Duke Math. J. vol. 4 (1938) pp. 664–667; J. Levitzki, *Solution of a problem of G. Köthe*, Amer. J. Math. vol. 67 (1945) pp. 437–442.

In connection with Theorem 2 it would be of interest to know if there exist L-nilpotent rings for which R_2 is not nilpotent. It would be enough to exhibit a ring R for which

$$(x \circ y) \circ z = 0 \qquad \text{for all } x,\ y,\ z \in R$$

and such that the subring generated by elements of the form $(x \circ y)$ is not nilpotent. The author has been unable to construct such a ring but it seems fairly safe to conjecture that such a one exists, and indeed with a countable generating set.

Since $R/\overline{R}_2$ is commutative and $\overline{R}_2$ is nilpotent we have at once from an earlier result of the author:[1]

THEOREM 3. *A finitely generated L-nilpotent ring is of finite class.*

Finally, it is clear that we have the following criterion for the nilpotency of a finitely generated nil-ring:

THEOREM 4. *A finitely generated nil-ring is nilpotent if and only if its associated Lie ring is nilpotent.*

This last theorem may be compared with Kaplansky's result on finitely generated nil-algebras,[4] which states that, provided the ground field has enough elements, such an algebra is nilpotent if and only if there exists a fixed integer ρ such that $x^\rho = 0$ for all elements x in the algebra. Our theorem shows that this condition may be replaced by the requirement that all commutators of a fixed weight vanish.

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[4] I. Kaplansky, *On a problem of Kurosch and Jacobson*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 496–500. *Added in proof.* In a recent paper (Bull. Amer. Math. Soc. vol. 52 (1946) pp. 1033–1035) J. Levitzki has proved a more general theorem to the effect that every finitely generated nil-ring of bounded index is nilpotent.