ON THE REDUCTION OF THE CONJUGATING REPRESENTATION OF A FINITE GROUP

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1. Introduction. By the conjugating representation P of a finite group G of order $g>1$, with elements γ_i, is meant the representation of G by permutation matrices $P(\gamma_i)$ such that

$$\gamma_i^{-1}\boldsymbol{\gamma}\gamma_i = P(\gamma_i)\boldsymbol{\gamma}. \tag{1}$$

Here we define the group vector $\boldsymbol{\gamma}$ to be a $g\times 1$ column vector whose entries are the elements of G, arranged so that the identity element γ_1 is first, and so that the h_σ elements of a class C_σ of conjugate elements are listed consecutively, forming a class vector $\boldsymbol{\gamma}_\sigma$ which is a subvector of the group vector $\boldsymbol{\gamma}$.

From a study of two different partial decompositions of the linear group P and its subsequent complete reduction into irreducible components Γ_ρ, the principal theorem is obtained, which relates the multiplicities of the irreducible components of the direct products $\overline{\Gamma}_\rho\times\Gamma_\rho$ with those of certain transitive constituents of P. Furthermore, a matrix T is described which completely and simultaneously reduces the right and left regular representations as well as the conjugating representation.

2. The transitive constituents of the conjugating representation. The $g\times g$ permutation matrices of the right and left regular representations, respectively, are defined by right or left multiplication of the group vector $\boldsymbol{\gamma}$ by a group element γ_i, thus:

$$\boldsymbol{\gamma}\gamma_i = R(\gamma_i)\boldsymbol{\gamma}, \qquad \gamma_i\boldsymbol{\gamma} = L(\gamma_i)\boldsymbol{\gamma}. \tag{2}$$

They form transitive groups of permutation matrices, one isomorphic and the other anti-isomorphic with G. The matrix $R(C_\sigma)$, obtained by summing the matrices $R(\gamma_i)$ over a class C_σ, is identical with the corresponding matrix $L(C_\sigma)$. Each matrix $R(\gamma_i)$ is permutable with every matrix $L(\gamma_j)$.

A group of permutation matrices, which we call the *conjugating representation* P of G, is defined by assigning to the group element γ_i the matrix $P(\gamma_i)$, where

$$P(\gamma_i) = L(\gamma_i^{-1})R(\gamma_i), \qquad P(\gamma_i)P(\gamma_j) = P(\gamma_i\gamma_j). \tag{3}$$

Presented to the Society, August 22, 1946; received by the editors November 7, 1946.

The matrices $P(\gamma_i)$ may also be defined directly by the equation (1).

From equation (1) it is apparent that the permutation group P is intransitive (for $g>1$), having one transitive constituent P corresponding to each of the r distinct classes C_σ. In terms of the class vector γ_σ we define the transitive "class representation" P_σ as follows:

$$\gamma_i^{-1}\gamma_\sigma\gamma_i = P_\sigma(\gamma_i)\gamma_\sigma. \tag{4}$$

It is well known[1] that in any transitive permutation group of degree n which is homomorphic with a given group G, there is a subgroup leaving a specified symbol fixed, to which corresponds a subgroup H of index n in G, and that the given permutation group is equivalent to the permutation group G_H on the right cosets $H\gamma_\alpha$ of G. To γ_i in G corresponds the permutation $H\gamma_\alpha \to H\gamma_\alpha\gamma_i$ in G_H. If K is the largest subgroup of H which is invariant in G, then the permutation group G_H is isomorphic with the factor group G/K.

For the transitive class representation P_σ, the subgroup H is the normalizer N_σ of a chosen element of the class C_σ. Since each N_σ contains the center C of G, each group P_σ is a representation of the factor group G/C. It is never a faithful (isomorphic) representation of G when the center contains more than one element.

Each of the groups P_σ, considered as a linear group, may be completely reduced by a change of basis into the direct sum of irreducible linear groups. Let $\mu_{\sigma\rho}$ be the multiplicity in P_σ of the irreducible component Γ_ρ and let Γ_1 be the identity representation. Then since P_σ is transitive, $\mu_{\sigma 1}=1$. In the complete reduction of the conjugating representation P, if μ_ρ is the multiplicity of the component Γ_ρ,

$$P \cong \sum_\rho \mu_\rho\Gamma_\rho, \qquad \text{where } \mu_\rho = \sum_{\sigma=1}^{r} \mu_{\sigma\rho}. \tag{5}$$

In particular, if the center of G contains more than one element, the coefficient μ_ρ is zero for every faithful representation Γ_ρ of G, and for all other representations which do not represent every invariant element by the unit matrix.

3. **Reduction by idempotents of the group algebra.** The right and left regular representations of the group G induce corresponding representations of the group algebra $\mathfrak{A}$ whose typical element $a=\sum a_i\gamma_i$ is a linear combination of group elements with coefficients from a specified field such as the field of complex numbers. It is known[2] that matrices T exist which transform the group vector γ into some new basis

[1] A. Speiser, *Die Theorie der Gruppen von endlicher Ordnung*, 3d ed., Berlin, 1937 (Dover, New York, 1945) p. 113.

[2] A. Speiser, loc. cit. p. 178.

vector $T^{-1}\gamma$, such that both the right and left regular representations are thereby completely and simultaneously reduced, and such that equivalent irreducible components of the right representation are actually identical with each other, and are merely transposes of the corresponding components in the anti-isomorphic left representation. The component matrices $\Gamma_\rho(\gamma_i)$ may further be assumed to be in unitary form, and we have

$$\Gamma_\rho'(\gamma_i^{-1}) = \overline{\Gamma}_\rho(\gamma_i). \tag{6}$$

In both of the transformed representations the class C_σ is represented by the same diagonal matrix $T^{-1}R(C_\sigma)T = T^{-1}L(C_\sigma)T$. The r classes C_σ of G are linearly independent in both regular representations, so these diagonal matrices must be linear combinations of r^* idempotent matrices I_ρ, where $r^* \geqq r$. By means of the primitive diagonal idempotents I_ρ we define the blocks R_ρ and L_ρ of which the transformed right and left representations are respectively the direct sums:

$$R_\rho(a) = I_\rho T^{-1}R(a)TI_\rho; \qquad L_\rho(a) = I_\rho T^{-1}L(a)TI_\rho. \tag{7}$$

Since two nonequivalent irreducible components must differ in their representations of at least one class C_σ, the irreducible components of the block $R_\rho(a)$ are all equivalent, and may each be written as the same $\Gamma_\rho(a)$ by suitable choice of T. Equivalent to these, but written in the transposed form $\Gamma_\rho'(a)$ to produce the required anti-isomorphism, are the components of the left block $L_\rho(a)$. Because the matrices of R_ρ and L_ρ commute with each other, the multiplicity of the component Γ_ρ in R_ρ must equal the degree of Γ_ρ' in L_ρ, namely n_ρ. Hence the diagonal idempotent I_ρ has n_ρ^2 1's and defines a subspace of the g-dimensional vector space in which $R_\rho(a)$ and $L_\rho(a)$ may be written as direct products (in opposite orders) of a unit matrix and a representation matrix each of degree n_ρ:

$$R_\rho(a) = \Gamma_\rho'(1) \times \Gamma_\rho(a); \qquad L_\rho(a) = \Gamma_\rho'(a) \times \Gamma_\rho(1). \tag{8}$$

Using the same suitably chosen T, whose coefficients we shall describe later, we transform the conjugating representation, defining

$$Q(\gamma_i) = T^{-1}P(\gamma_i)T. \tag{9}$$

From equations (9), (3), (7), (8), and (6), it then follows that the block Q_ρ of Q defined by the idempotent I_ρ has the form

$$Q_\rho(\gamma_i) = I_\rho Q(\gamma_i)I_\rho = \overline{\Gamma}_\rho(\gamma_i) \times \Gamma_\rho(\gamma_i), \tag{10}$$

and is thus the direct product of the two conjugate imaginary irreducible representations $\bar{\Gamma}_\rho$ and Γ_ρ of the group G. Consequently the multiplicity of any irreducible representation as a component of the conjugating representation P is the sum of its multiplicities in the r^* direct products $\bar{\Gamma}_\rho \times \Gamma_\rho$.

4. **The balancing of multiplicities.** Combining the result of §§2 and 3 we obtain our principal theorem:

THEOREM 1. *The sum of the multiplicities of a given irreducible representation Γ_ρ as a component in the direct products $\bar{\Gamma}_\tau \times \Gamma_\tau$ is equal to the sum of its multiplicities in the transitive class representations P_σ (permutations on cosets with respect to a normalizer).*

Applying Theorem 1 to the component Γ_1 which occurs just once in each $\bar{\Gamma}_\tau \times \Gamma_\tau$ and P_σ, we obtain the following well known result.

COROLLARY 1. *The number of nonequivalent irreducible representations of a finite group is equal to the number of its classes.*

To illustrate Theorem 1, we give below the two decompositions of the conjugating representation of the symmetric group of order 24. This group has five nonequivalent irreducible representations, of degrees $n_\rho = 1, 1, 3, 3, 2$, respectively, and five class representations P_σ, of degrees $h_\sigma = 1, 3, 8, 6, 6$, respectively. The decomposition of each

	$\bar{\Gamma}_\tau \times \Gamma_\tau$					P		P_σ				
τ:	1	2	3	4	5		σ:	1	2	3	4	5
n_τ^2:	1	1	9	9	4	$g=24$	h_σ:	1	3	8	6	6
Γ_1:	1	1	1	1	1	$\mu_1=5$		1	1	1	1	1
Γ_2:					1	$\mu_2=1$				1		
Γ_3:			1	1		$\mu_3=2$				1	1	
Γ_4:			1	1		$\mu_4=2$				1		1
Γ_5:			1	1	1	$\mu_5=3$			1		1	1

Table of multiplicities

of the five direct products $\bar{\Gamma}_\tau \times \Gamma_\tau$ is given in one of the left-hand columns, that of the conjugating representation P in the center column, and those of the class representations P_σ in the right-hand columns.

From Theorem 1 and the discussion of §2, we obtain also the following theorem.

THEOREM 2. *If Γ is an irreducible representation of a finite group G whose center C contains more than one element, then each irreducible* *component of* $\bar{\Gamma}\times\Gamma$ *is a representation of* G/C, *and is not a faithful (isomorphic) representation of* G.

One illustration of Theorem 2 is afforded by considering a representation Γ of the quaternion group which is irreducible over the field of complex numbers. Since the center C contains two elements, no component of $\bar{\Gamma}\times\Gamma$ is a faithful representation of the quaternion group. Similarly let us consider a group G which is represented faithfully by a group Γ of unitary symplectic matrices, irreducible over the field of complex numbers. Weyl[3] applies the term *symplectic* to matrices of degree $n=2\nu$ having an alternating bilinear invariant. In terms of real ν-dimensional matrices A, B, C, D and the corresponding unit matrix I such unitary symplectic matrices M_n and their invariant j may be written in the form

$$M_n = \begin{pmatrix} A+Bi & C+Di \\ -C+Di & A-Bi \end{pmatrix}, \qquad j = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{11}$$

This is equivalent to a set of ν-dimensional matrices $M_\nu = A+Bi+Cj+Dij$ having quaternion coefficients, and such that the inverse matrix is the transposed quaternion conjugate $A'-B'i-C'j-D'ij$. The group Γ is equivalent to $\bar{\Gamma}$. The existence in G of an invariant element of order 2 implies by Theorem 2 that $\bar{\Gamma}\times\Gamma$ is not a faithful representation of G.

5. **The reducing transformation.** It is known that a linear transformation with matrix T exists which completely reduces the right and left regular representations.[2] It is possible to find all such reducing matrices T in a fairly straightforward manner by making use of the conjugating representation.

We first define a $g\times g$ "entry matrix" $Z=\|Z_{ij}\|$ by assigning to the ith row in a specified order the $g=\sum n_\rho^2$ linearly independent entries (or coefficients) for the group element γ_i in a complete set of r nonequivalent irreducible unitary representations Γ_ρ. It is best to order these g entries first by representations Γ_ρ (Γ_1 leading), then by rows within the particular n_ρ-dimensional matrix of Γ_ρ, and finally by columns in Γ_ρ. The ordering of rows in Z shall be that of the group vector γ.

Then the matrix $P(\gamma_i)Z$ is a matrix similar to Z but with each group element which defines a row in Z replaced by its transform under γ_i, so that the rows of the matrix Z are permuted by $P(\gamma_i)$. The same reordering of the coefficients of Z could have been obtained by postmultiplying Z by the matrix $Q(\gamma_i)$ of (9), which is

[3] H. Weyl, *The classical groups*, Princeton, 1939, p. 165 ff.

partially decomposed according to the idempotents I_ρ. For ZI_ρ is a rectangular matrix of g rows and n_ρ^2 columns, whose coefficients are transformed by a direct product matrix $\overline{\Gamma}_\rho(\gamma_i)\times\Gamma_\rho(\gamma_i)$ applied on the right just as if each individual group element were transformed by γ_i. Hence using (9), we have

$$P(\gamma_i)Z = ZQ(\gamma_i) = ZT^{-1}P(\gamma_i)T. \tag{12}$$

A similar argument shows that $R(\gamma_i)ZI_\rho$ and $L(\gamma_i)ZI_\rho$ are matrices like ZI_ρ in which the coefficients corresponding to the group element γ_j are replaced by those of $\gamma_j\gamma_i$ or of $\gamma_i\gamma_j$ respectively. This however is the same as either $ZI_\rho R_\rho(\gamma_i)$ or $ZI_\rho L_\rho(\gamma_i)$, where R_ρ and L_ρ are defined in (7). Summing over ρ we have

$$R(\gamma_i)Z = ZT^{-1}R(\gamma_i)T; \qquad L(\gamma_i)Z = ZT^{-1}L(\gamma_i)T. \tag{13}$$

Writing $T=ZV$ in (12) and (13) we see that the nonsingular matrix V permutes with each of the matrices $T^{-1}R(\gamma_i)T$ and $T^{-1}L(\gamma_i)T$ and their product $Q(\gamma_i)$. Thus,

$$\begin{aligned} V[T^{-1}R(\gamma_i)T] &= V(ZV)^{-1}R(\gamma_i)ZV = V(ZV)^{-1}ZT^{-1}R(\gamma_i)TV \\ &= [T^{-1}R(\gamma_i)T]V, \\ V[T^{-1}L(\gamma_i)T] &= T^{-1}L(\gamma_i)TV, \\ VQ(\gamma_i) &= Q(\gamma_i)V. \end{aligned} \tag{14}$$

These relations (14) are possible for all γ_i if and only if V lies in the intersection of the commutators of $T^{-1}R(a)T$ and $T^{-1}L(a)T$. Hence V is a nonsingular linear combination of the idempotents I_ρ.

Now the well known orthogonality relations for the coefficients in the irreducible group representations imply that

$$\overline{Z}'Z = \sum_\rho (g/n_\rho)I_\rho. \tag{15}$$

It follows that the matrix $T=ZV$ will be unitary if and only if

$$V = \sum_\rho \omega_\rho(n_\rho/g)^{1/2}I_\rho, \qquad \text{where } \bar{\omega}_\rho\omega_\rho = 1. \tag{16}$$

A convenient choice is to take $\omega_\rho=1$.

THEOREM 3. *A unitary matrix T which completely reduces the right and left representations and the conjugating representation may be formed by multiplying each element of the entry matrix Z described above by the appropriate factor $(n_\rho/g)^{1/2}$, where n_ρ is the degree of the irreducible representation associated with the particular column of Z.*

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