$$\sum{}' f^{(1)}(n_1)f^{(2)}(n_2)\cdots f^{(\nu)}(n_\nu) = (1+o(1))Dn^{\nu-1},$$

also

$$\sum_{m=1}^{n} f^{(1)}(m+k_1)f^{(2)}(m+k_2)\cdots f^{(\nu)}(m+k_\nu) = (1+o(1))En,$$

D and E are given by a complicated expression.

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ON A CLASS OF TAYLOR SERIES

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1. **Introduction.** Consider the Taylor series $\sum_{n=0}^{\infty} a_n z^n$. Suppose that the singularities of the function defined by the series all lie in certain regions of the complex plane and that the coefficients possess certain arithmetical properties. Mandelbrojt[1] has shown that under restrictions of this nature it is possible to predict the form of the function defined by the series. This note is concerned with the establishing of a new method to obtain more general results of this nature.

2. **The method.** The method that is employed here is an adaptation of a method used by Lindelöf [2] in the problem of representation of a function defined by a series.

Let $f(z)$ be regular in a region D of the complex plane. Suppose that there exists a linear transformation $t=h(z)$ which maps the region of regularity into a region which includes the unit circle of the t-plane in its interior. Let $z=g(t)$ be the inverse of this transformation. Then $F(t) \coloneqq f(g(t))$ is regular in this region in the t-plane. For this note it is convenient to suppose that $z=0$ corresponds to $t=0$ in the mapping. We may expand $g(t)$ in a Taylor series about $t=0$ and obtain

$$z = b_1 t + b_2 t^2 + \cdots \tag{2.1}$$

convergent for t in absolute value sufficiently small. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2.2}$$

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[1] See, Mandelbrojt [3]. Numbers in brackets refer to the bibliography at the end of the paper.

be the element of $f(z)$ at the origin. For t in absolute value sufficiently small we may substitute (2.1) in (2.2) and obtain

$$F(t) = f(g(t)) = \sum_{n=0}^{\infty} C_n t^n. \tag{2.3}$$

We have seen, however, that $F(t) = f(g(t))$ is regular in a region in the t-plane which includes the unit circle in its interior. Hence the radius of convergence of (2.3) is greater than one. Therefore we may write

$$\limsup_{n\to\infty} (|C_n|)^{1/n} < 1. \tag{2.4}$$

As the C_n are polynomial combinations of the a_n and b_n we see that under certain circumstances (2.4) may imply $C_n = 0$ for n greater than some n_0. For example, if the C_n are all integers (2.4) implies the existence of an n_0 such that $C_n = 0$ for $n > n_0$. It is also clear that if $C_n = C_n' + iC_n''$ where C_n' and C_n'' are integers that the conclusion $C_n = 0$, $n > n_0$, still holds. Under these circumstances we obtain upon substituting $t = h(z)$ in (2.3)

$$f(z) = \sum_{n=0}^{n_0} C_n [h(z)]^n.$$

3. **Applications.** We now proceed to the proof of the principal theorem.

THEOREM 3.1. *If the series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has rational coefficients such that there exists an integer L for which the quantities*

$$a_0,\ a_1 L,\ a_2 L^2,\ \cdots,\ a_n L^n, \cdots$$

are integers, and if the function defined by the series is regular exterior to and on the circumference of a circle with center $(LK'/(K'^2+K''^2-1), LK''/(K'^2+K''^2-1))$ and radius $L/(K'^2+K''^2-1)$ where K' and K'' are integers, $K'^2+K''^2 \neq 1$, then the function defined by the series is of the form

$$\frac{P(z)}{(L/(K' + iK'') - z)^{n_0}}$$

where $P(z)$ is a polynomial and n_0 is a positive integer.

It is easily shown that the transformation

$$t = \frac{z}{L - (K' + iK'')z} \tag{3.1}$$

maps the region of regularity into a region containing the unit circle of the t-plane in its interior. By substituting the solution of (3.1) for z in (2.2) and setting $K=K'+iK''$ we have

$$F(t) = f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{L^n t^n}{(1+Kt)^n} = \sum_{n=0}^{\infty} a_n L^n t^n \sum_{m=0}^{\infty} C_{-n,m}(Kt)^m.$$

Then since the series are absolutely convergent, we obtain

$$F(t) = \sum_{n=0}^{\infty} [a_n L^n C_{-n,0} + C_{-n,1} K L^{n-1} a_{n-1} + \cdots + a_0 C_{-n,n} K^n] t^n,$$

$$(3.2)\quad F(t) = \sum_{n=0}^{\infty} C_n t^n.$$

Here $C_n = C_n' + iC_n''$ where C_n' and C_n'' are integers, for the $C_{-n,m}$ are binomial coefficients and $a_n L^n$ is an integer for all $n \geqq 0$ by hypothesis. Also K' and K'' are integers. Hence, from the discussion in §2 it follows that

$$\limsup_{n\to\infty} |C_n|^{1/n} < 1$$

implies the existence of a numbei n_0 such that $C_n=0$ for $n>n_0$. Therefore upon substituting (3.1) in (3.2) we have

$$f(z) = \sum_{n=0}^{n_0} C_n \left(\frac{z}{L-Kz}\right)^n = \frac{P(z)}{(L/(K'+iK'')-z)^{n_0}}.$$

This completes the proof of the theorem. If now we choose $K''=0$ and $K'=L$ we have the theorem of Mandelbrojt [3]. This proof of Mandelbrojt's theorem has some points in common with a proof of the same theorem due to Achyser [1].[2] If in addition the a_n are all integers we may set $L=1$ and have a new theorem.

We now proceed to the proof of the following theorem.

THEOREM 4.1. *If the series* $\sum_{n=0}^{\infty} a_n z^n$ *has rational coefficients such that there exists an integer L for which the quantities*

$$a_0,\ a_1 L,\ a_2 L^2,\ \cdots,\ a_n L^n,\ \cdots$$

[2] The author is indebted to the referee for this observation.

are integers and if the function defined by the series is regular in the half-plane $R(z) \leqq L/2$ including the point at infinity then the series defines a function of the form

$$\frac{P(z)}{(L-z)^{n_0}}$$

where $P(z)$ is a polynomial and n_0 is a positive integer.

From the hypothesis it is easily seen that the transformation

$$t = \frac{z}{L-z} \tag{3.3}$$

maps the region of regularity into a region which includes the unit circle in the t-plane in its interior. Upon solving (3.3) for z and substituting in (2.2) we obtain

$$\begin{aligned} F(t) &= f\left(\frac{Lt}{1+t}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{Lt}{1+t}\right)^n \\ &= \sum_{n=0}^{\infty} a_n t^n L^n \sum_{m=0}^{\infty} C_{-n,m}(t)^m = \sum_{n=0}^{\infty} C_n t^n. \end{aligned} \tag{3.4}$$

Then by the same arguments employed in Theorem (3.1) it follows that there exists an n_0 such that $C_n = 0$ for $n > n_0$. Therefore by substituting from (3.3) in (3.4) we have

$$f(z) = \sum_{n=0}^{n_0} C_n \left(\frac{z}{L-z}\right)^n = \frac{P(z)}{(L-z)^{n_0}},$$

where $P(z)$ is a polynomial and n_0 is a positive integer. This completes the proof of the theorem.

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