MEAN VALUES OF PERIODIC FUNCTIONS

PAUL CIVIN

Let L^p denote the class of complex measurable functions of period 2π for which $M_p(f) < \infty$, where

$$M_p(f) = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p} \qquad (1 \leqq p < \infty), \tag{1}$$

$$M_\infty(f) = \operatorname*{ess.\,sup}_{0 \leqq x \leqq 2\pi} |f(x)|. \tag{2}$$

Let $K_{m,p}$ denote the subset of L^p whose elements, $f(x)$, have a Fourier series of the form

$$\sum_{n=m}^{\infty} (a_n \cos nx + b_n \sin nx) \qquad (m \geqq 1). \tag{3}$$

The functions of $K_{m,p}$ and their Fourier series (3) are transformed by the real number δ and the sequence of real numbers $\lambda = \{\lambda(n)\}$ into the series

$$\begin{aligned} &\sum_{n=m}^{\infty} \lambda(n) \left\{a_n \cos\left(nx + \frac{\delta\pi}{2}\right) + b_n \sin\left(nx + \frac{\delta\pi}{2}\right)\right\} \\ &= \sum_{n=m}^{\infty} \lambda(n) \left\{\left(a_n \cos\frac{\delta\pi}{2} + b_n \sin\frac{\delta\pi}{2}\right)\cos nx \right. \\ &\qquad \left. + \left(b_n \cos\frac{\delta\pi}{2} - a_n \sin\frac{\delta\pi}{2}\right)\sin nx\right\}. \end{aligned} \tag{4}$$

A slight modification of the well known result[1] [5, pp. 100 ff.][2] for the case in which $\delta = 0$ shows that if

$$\sum_{n=m}^{\infty} \lambda(n) \cos\left(nx - \frac{\delta\pi}{2}\right) = \sum_{n=m}^{\infty} \lambda(n) \left\{\cos\frac{\delta\pi}{2}\cos nx + \sin\frac{\delta\pi}{2}\sin nx\right\} \tag{5}$$

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[1] Although the convention is adopted in *Trigonometrical series* that $f(x)$ is real, the results of the sections of *Trigonometrical series* to which reference is made in this note hold for complex $f(x)$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

is a Fourier or even a Fourier-Stieltjes series then (4) is the Fourier series of a function of L^p, $1 \leqq p \leqq \infty$. Throughout the sequel it is assumed that (5) is a Fourier series. Series (4) is therefore the Fourier series of a function $g(x) \in L^p$ and is of the form of (3), hence $g(x) \in K_{m,p}$. The transformation determined by the number δ and the sequence λ is thus a transformation of $K_{m,p}$ onto itself.

The objective of the present note is to establish an inequality between the means $M_p(f)$ and $M_p(g)$ which holds for all $f(x) \in K_{m,p}$. For the essentially bounded case this has been done by B. v. Sz. Nagy [3], and for completeness the result is stated as a lemma.

LEMMA 1 [Sz. Nagy]. *If $f(x) \in K_{m,\infty}$, then* (4) *is the Fourier series of a continuous function $g(x) \in K_{m,\infty}$ and*

$$M_\infty(g) \leqq A(\lambda, \delta, m) M_\infty(f), \tag{6}$$

where $A(\lambda, \delta, m)$ is a function only of the indicated variables and not of the particular $f(x) \in K_{m,\infty}$.

The notation $A(\lambda, \delta, m)$ will be used throughout the sequel to denote the smallest possible function which will satisfy (6) for all $f(x) \in K_{m,\infty}$.

LEMMA 2. *If $f(x) \in K_{m,2}$, then* (4) *is the Fourier series of a function $g(x) \in K_{m,2}$ and*

$$M_2(g) \leqq \Lambda(m) M_2(f), \tag{7}$$

where $\Lambda(m) = \max_{m \leqq n} |\lambda(n)|$.

The Riesz-Fischer theorem asserts that (4) is the Fourier series of a function $g(x) \in L^2$ and that

$$\begin{aligned} M_2(g) &= \left\{\pi \sum_{n=m}^{\infty} (\lambda(n))^2 (|a_n|^2 + |b_n|^2)\right\}^{1/2} \\ &\leqq \Lambda(m) \left\{\pi \sum_{n=m}^{\infty} (|a_n|^2 + |b_n|^2)\right\}^{1/2} = \Lambda(m) M_2(f). \end{aligned}$$

If $\Lambda(m) = |\lambda(r)|$ $(r \geqq m)$, then $f(x) = \cos rx$ gives equality in (7).

It is now possible to state the principal theorem.

THEOREM 1. *If $f(x) \in K_{m,p}$, then* (4) *is the Fourier series of a function $g(x) \in K_{m,p}$ and*

$$M_p(g) \leqq \Lambda^{2/p}(m) A^{(p-2)/p}(\lambda, \delta, m) M_p(f) \qquad (2 \leqq p < \infty), \tag{8}$$

$$M_p(g) \leqq B_p \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_p(f) \qquad (1 < p \leqq 2) \tag{9}$$

where $\Lambda(m)$ *and* $A(\lambda, \delta, m)$ *are defined in Lemmas* 1 *and* 2, B_p *is a constant depending only on* p *and not on the particular* $f(x)\in K_{m,p}$, *and* $p'=p/(p-1)$.

The transformation of $f(x)$ into $g(x)$, or of series (3) into series (4), is a linear transformation of $K_{m,2}$ onto itself, and also of $K_{m,\infty}$ onto itself. The direct application of an interpolation scheme for L^p fails in the attempt to establish (8) since the space $K_{m,p}$ is a nondense linear subspace of L^p. However, the proof of the interpolation result for L^p as given in *Trigonometrical series* [5, p. 198 ff.] carries through for the space $K_{m,p}$ on the basis of the following lemma.

LEMMA 3. *The step functions of* $K_{m,p}$ *are dense in* $K_{m,p}$ *in the metric of* L^p $(1<p<\infty)$.

The step functions of $K_{m,p}$ are those functions of $K_{m,p}$ which assume only a finite number of values and assume each of these values on a finite sum of intervals in $(0, 2\pi)$. Suppose $f(x)\in K_{m,p}$ and η is a positive number. The density of the continuous functions of L^p requires the existence of a continuous function $h(x)\sim c_0/2+\sum_{n=1}^{\infty}(c_n \cos nx+d_n \sin nx)$ such that $M_p(f-h)<\eta$ and $|c_0|/2+\sum_{n=1}^{m-1}(|c_n|+|d_n|)<\eta$. The function $k(x)=h(x)-c_0/2-\sum_{n=1}^{m-1}(c_n \cos nx+d_n \sin nx)$ is therefore a continuous function of class $K_{m,p}$ and $M_p(f-k)\leqq M_p(f-h)+\eta(2\pi)^{1/p}<8\eta$. Hence the continuous functions of $K_{m,p}$ are dense in $K_{m,p}$. It is sufficient therefore to show that the continuous functions of L^p can be approximated uniformly by step functions of $K_{m,p}$.

Consider first a continuous $f(x)\in K_{1,p}$. For any positive η, there is a step function $s(x)$ such that $|f(x)-s(x)|\leqq\eta$ for all x. If $c=(1/2\pi)\int_0^{2\pi}s(x)dx$, then since $\int_0^{2\pi}f(x)dx=0$, $|c|\leqq(1/2\pi)\int_0^{2\pi}|s(x)-f(x)|dx\leqq\eta$. The step function $t(x)=s(x)-c$ is therefore in $K_{1,p}$ and $|f(x)-t(x)|\leqq 2\eta$.

Suppose next that it has been demonstrated that the continuous functions of $K_{r,p}$ can be uniformly approximated by step functions of $K_{r,p}$ for $1\leqq r<m$. Since $K_{r,p}\supset K_{s,p}$ if $r<s$, any continuous function of $K_{m,p}$ can be uniformly approximated by step functions whose Fourier coefficients of order less than $(m-1)$ vanish. Hence if $f(x)$ is a continuous function of $K_{m,p}$ and η is a positive number, there is a step function $s(x)\in K_{m-1,p}$ such that $|f(x)-s(x)|<\eta$ for all x. Suppose that $c=(1/4)\int_0^{2\pi}s(x)\cos(m-1)xdx$ and $d=(1/4)\int_0^{2\pi}s(x)\sin(m-1)xdx$. Since $f(x)\in K_{m,p}$, both $|c|<2\eta$ and $|d|<2\eta$. Suppose the function $t(x)=s(x)-c \operatorname{sgn}\cos(m-1)x-d\operatorname{sgn}\sin(m-1)x$, where $\operatorname{sgn} u=0$ if $u=0$ and $\operatorname{sgn} u=u/|u|$ if $u\neq 0$. It can be shown by direct calculation that the step function $t(x)\in K_{m,p}$. Since $|f(x)-t(x)|\leqq|f(x)-s(x)|$

$+|c \operatorname{sgn} \cos (m-1)x| + |d \operatorname{sgn} \sin (m-1)x| < 5\eta$, the function $t(x)$ gives the desired uniform approximation.

In order to establish (9), it is first noted [5, p. 105] that since $g(x) \in L^p$,

$$M_p(g) = \sup \left| \int_0^{2\pi} g(x)\overline{h(x)}dx \right| \tag{10}$$

with the supremum taken over all $h(x)$ for which $M_{p'}(h) \leqq 1$. Hence if η is a positive number there is an $h(x)$ for which

$$M_{p'}(h) \leqq 1$$

and

$$M_p(g) - \eta \leqq \left| \int_0^{2\pi} g(x)\overline{h(x)}dx \right| . \tag{11}$$

Suppose that $h(x) \sim r_0/2 + \sum_{n=1}^{\infty}(r_n \cos nx + s_n \sin nx)$ and that $h_m(x) \sim \sum_{n=m}^{\infty}(r_n \cos nx + s_n \sin nx)$. A double application of Parseval's relation for functions of L^p and $L^{p'}$ shows that

$$\begin{aligned} \int_0^{2\pi} g(x)\overline{h(x)}dx &= \pi \sum_{n=m}^{\infty} \left\{ \lambda(n) \left(a_n \cos\frac{\delta\pi}{2} + b_n \sin\frac{\delta\pi}{2} \right) \overline{r_n} \right. \\ &\qquad \left. + \lambda(n) \left(b_n \cos\frac{\delta\pi}{2} - a_n \sin\frac{\delta\pi}{2} \right) \overline{s_n} \right\} \\ &= \pi \sum_{n=m}^{\infty} \left\{ \lambda(n) \left(\overline{r_n} \cos\frac{\delta\pi}{2} - \overline{s_n} \sin\frac{\delta\pi}{2} \right) a_n \right. \\ &\qquad \left. + \lambda(n) \left(\overline{s_n} \cos\frac{\delta\pi}{2} + \overline{r_n} \sin\frac{\delta\pi}{2} \right) b_n \right\} \\ &= \int_0^{2\pi} \overline{H(x)} f(x)dx, \end{aligned} \tag{12}$$

where

$$\begin{aligned} H(x) &\sim \sum_{n=m}^{\infty} \lambda(n) \left\{ \left(r_n \cos\frac{\delta\pi}{2} - s_n \sin\frac{\delta\pi}{2} \right) \cos nx \right. \\ &\qquad \left. + \left(s_n \cos\frac{\delta\pi}{2} + r_n \sin\frac{\delta\pi}{2} \right) \sin nx \right\} \\ &= \sum_{n=m}^{\infty} \lambda(n) \left\{ r_n \cos\left(nx - \frac{\delta\pi}{2} \right) + s_n \sin\left(nx - \frac{\delta\pi}{2} \right) \right\} . \end{aligned}$$

Thus $H(x)$ is the transform of $h_m(x)$ which is obtained by use of the number $-\delta$ and the sequence λ. The application of Hölder's inequality followed by the use of (8) then shows that

$$\left| \int_0^{2\pi} \overline{H(x)} f(x) dx \right| \leqq M_{p'}(H) M_p(f) \tag{13}$$

$$\leqq M_p(f) \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_{p'}(h_m).$$

A well known result of M. Riesz [1] implies that

$$M_{p'}(h_m) \leqq B_p M_{p'}(h) \tag{14}$$

where B_p depends only on p and not on the functions involved. The combination of formulas (10) through (14) then shows that

$$M_p(g) - \eta \leqq B_p \Lambda^{2/p'}(m) A^{(p'-2)/p'}(\lambda, -\delta, m) M_p(f)$$

and (9) follows since η was arbitrary.

The result of Theorem 1 will now be applied to integrals of functions of $K_{m,p}$. It is convenient to use the definition of the integral of order α which is due to Weyl [4]. For any positive α, for $f(x) \in K_{m,p}$ and with Fourier series (3), the integral of order α, $f_\alpha(x)$, is defined as

$$f_\alpha(x) = \sum_{n=m}^{\infty} \frac{1}{n^\alpha} \left\{ a_n \cos\left(nx - \frac{\alpha\pi}{2}\right) + b_n \sin\left(nx - \frac{\alpha\pi}{2}\right) \right\}.$$

Thus $f_\alpha(x)$ is the transform of $f(x)$ of the type of (4) with $\delta = -\alpha$ and the sequence $\{\lambda(n)\} = \{n^{-\alpha}\}$. Various results are known concerning the relationship between $M_\infty(f_\alpha)$ and $M_\infty(f)$, the most inclusive of which is that of Sz. Nagy [3] who shows that

$$\begin{aligned} A(\lambda, \delta, m) &= A(\{n^{-\alpha}\}, -\alpha, m) \\ &\leqq (4/\pi m^\alpha) \left\{ \left| \cos \frac{\alpha\pi}{2} \right| \sum_{v=0}^{\infty} (-1)^v (2v+1)^{-(1+\alpha)} \right. \\ &\qquad \left. + \left| \sin \frac{\alpha\pi}{2} \right| \sum_{v=0}^{\infty} (2v+1)^{-(1+\alpha)} \right\} \\ &\leqq (4/\pi m^\alpha) H(\alpha). \end{aligned}$$

It can also be seen from [3] that

$$A(\{n^{-\alpha}\}, \alpha, n) \leqq (4/\pi m^\alpha) H(\alpha).$$

A direct application of Theorem 1 yields the following theorem.

THEOREM 2. *If $f(x) \in K_{m,p}$, and $f_\alpha(x)$ is its integral of order α (α not necessarily integral) then*

$$
(15)\qquad \begin{aligned} M_p(f_\alpha) &\leqq m^{-\alpha}(4H(\alpha)/\pi)^{(p-2)/p}M_p(f) && (2 \leqq p),\\ M_p(f_\alpha) &\leqq B_p m^{-\alpha}(4H(\alpha)/\pi)^{(2-p)/p}M_p(f) && (1 < p \leqq 2)\end{aligned}
$$

where B_p *is the constant of Theorem* 1 *and*

$$
H(\alpha) = \left|\cos\frac{\alpha\pi}{2}\right| \sum_{v=0}^{\infty} (-1)^v(2v+1)^{-(1+\alpha)} + \left|\sin\frac{\alpha\pi}{2}\right| \sum_{v=0}^{\infty} (2v+1)^{-(1+\alpha)}.
$$

A result of Schmidt [2] shows that for the real functions of $K_{1,p}$ and α integral the coefficient of $M_p(f)$ in (15) is not the best possible.

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University of Oregon