ON THE NON-SUMMABILITY $(C, 1)$ OF FOURIER SERIES

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Let

$$\phi(t) \sim \sum_{n=1}^{\infty} a_n \cos nt, \qquad \phi(0) = 0, \tag{1}$$

be the Fourier series associated with an even function $\phi(t)$ which is integrable (L) over the interval $(0, \pi)$ and defined outside this range by periodicity.

Lebesgue [**2**, pp. 561–562][1] proved that the series (1) is summable $(C, 1)$ to zero at the point $t=0$ if, as $t \to 0$,

$$\int_0^t |\phi(t)| dt = o(t). \tag{2}$$

If, however, a condition weaker than (2) is satisfied, namely

$$\int_0^t \phi(t) dt = o(t), \tag{3}$$

as $t \to 0$, the series (1) is not necessarily summable $(C, 1)$. Hahn [**1**] gave an example to prove that the series (1) is not summable $(C, 1)$, though the condition (3) is satisfied but not (2). Prasad [**3**] has investigated whether, at the point $t=0$, the series (1) would be summable $(C, 1)$ if a condition stronger than (3) is satisfied, namely

$$\int_0^\delta \frac{\phi(t)}{t} dt = \lim_{t \to 0} \int_t^\delta \frac{\phi(t)}{t} dt = s, \tag{4}$$

say, exists as a non-absolutely convergent integral, and he has constructed an example to show that even when (4) is satisfied the series (1) is not necessarily summable $(C, 1)$. As the condition (4) implies (3), Prasad's example includes that of Hahn.

The object of the present note is to construct an example to prove that the series (1) is not necessarily summable $(C, 1)$ even though the condition

$$\phi_1(t) \equiv \int_t^\pi \frac{\phi(t)}{t} dt - s = o\left(1/\log\log\frac{1}{t}\right), \tag{5}$$

as $t \to 0$, is satisfied.

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[1] Numbers in brackets refer to the references cited at the end of the paper.

In view of the fact that the condition (5) is more stringent than (4) and (3), this example would be more far-reaching than those of Prasad and Hahn.

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EXAMPLE. Let

$$\lambda_r = 3^{r^2}, \qquad t_r = \pi/\lambda_r, \qquad t_0 = \pi, \qquad c_r = 1/(2r-1)^{1/2},$$

for $r=1, 2, 3, \cdots$. We define an even function $\phi(t)$ in $(0, \pi)$ by

$$\phi(0) = 0, \qquad \phi(t) = -c_r\lambda_r t\cos\lambda_r t, \tag{6}$$

for $t_r < t \leqq t_{r-1}$ and by periodicity outside. Then

$$\begin{aligned}\int_{t_r}^{t_{r-1}} |\phi(t)|\,dt &= c_r\lambda_r\int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} t\,|\cos\lambda_r t|\,dt \\ &= \frac{c_r}{\lambda_r}\int_{\pi}^{3^{2r-1}\pi} u\,|\cos u|\,du \\ &= O\left(\frac{3^{4r-2}}{\lambda_r}\right) = O\left(\frac{1}{3^{r^2-4r+2}}\right).\end{aligned}$$

Hence

$$\begin{aligned}\int_0^{\pi} |\phi(t)|\,dt &= O(1) + \sum_{r=4}^{\infty}\int_{t_r}^{t_{r-1}} |\phi(t)|\,dt \\ &= O(1) + \sum_{r=4}^{\infty} O\left(\frac{1}{3^{r^2-4r+2}}\right) \\ &= O(1),\end{aligned}$$

so that $\phi(t)$ is integrable (L) in $(0, \pi)$. Again

$$\int_{t_r}^{t_{r-1}} \frac{\phi(t)}{t}\,dt = -c_r\lambda_r\int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} \cos\lambda_r t\,dt = -c_r\left[\sin\lambda_r t\right]_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} = 0.$$

For $t_r < t < t_{r-1}$, we have

$$\int_t^{t_{r-1}} \frac{\phi(t)}{t}\,dt = -c_r\left[\sin\lambda_r t\right]_t^{\pi/\lambda_{r-1}} = c_r\sin\lambda_r t,$$

so that, taking $s=0$, we have, for $t_r < t < t_{r-1}$,

$$\phi_1(t) = \int_t^{\pi} \frac{\phi(t)}{t}\,dt = \int_t^{t_{r-1}} \frac{\phi(t)}{t}\,dt = c_r\sin\lambda_r t.$$

Now

$$\left|\log\log\frac{1}{t}\,\phi_1(t)\right| \leqq c_r \log\log\frac{1}{t_r} = c_r\log\log\frac{\lambda_r}{\pi}$$
$$= \frac{1}{(2r-1)^{1/2}}\log\log\frac{3^{r^2}}{\pi}$$
$$\leqq \frac{1}{(2r-1)^{1/2}}\,[2\log r + \log\log 3],$$

and hence, as $t\to 0$,

$$\phi_1(t) = o\left(1/\log\log\frac{1}{t}\right). \tag{7}$$

Now the necessary and sufficient condition that the Fourier series (1) be summable $(C, 1)$ is

$$\frac{1}{n}\int_0^{\pi}\phi(t)\sin^2\frac{nt}{2}\cdot\frac{dt}{t^2} = o(1), \qquad \text{as } n\to\infty,$$

or,

$$\frac{1}{n}\left[-\sin^2\frac{nt}{2}\cdot\frac{1}{t}\int_t^{\pi}\frac{\phi(t)}{t}\,dt\right]_0^{\pi}$$
$$+\frac{1}{2}\int_0^{\pi}\phi_1(t)\left[\frac{\sin nt}{t}-\frac{2}{n}\,\frac{\sin^2(nt/2)}{t^2}\right]dt = o(1).$$

This is $I_n+J_n=o(1)$, with

$$I_n = \int_0^{\pi}\phi_1(t)\,\frac{\sin nt}{t}\,dt,$$

$$J_n = \frac{1}{n}\int_0^{\pi}\phi_1(t)\left(\frac{\sin(nt/2)}{t}\right)^2 dt.$$

Also

$$J_n = \frac{1}{n}\int_0^{\pi/n} o\left(1/\log\log\frac{1}{t}\right)dt + \frac{1}{n}\int_{\pi/n}^{\pi} o\left(1/\log\log\frac{1}{t}\right)\frac{dt}{t^2}$$
$$= o(1) + \frac{1}{n}\int_{\pi/n}^{\pi} o\left(\frac{1}{t^2}\right)dt = o(1),$$

if (5) is satisfied.

Hence in order to show that the Fourier series associated with the function $\phi(t)$ defined by (6) is not summable $(C,1)$ at $t=0$, although the condition (7) is satisfied, we have only to prove that the integral

$$I_n = \int_0^\pi \phi_1(t)\,\frac{\sin nt}{t}\,dt$$

tends to ∞ as n assumes successively the values of integers

$$n = \lambda_r = 3^{r^2}, \qquad r = 1, 2, 3, \cdots,$$

that is

$$I_{\lambda_r} \to \infty, \qquad \text{as } r \to \infty.$$

Now

$$\begin{aligned} I_{\lambda_r} &= \int_0^\pi \phi_1(t)\,\frac{\sin \lambda_r t}{t}\,dt \\ &= \left[\int_0^{t_r} + \int_{t_r}^{t_{r-1}} + \sum_{i=1}^{r-1}\int_{t_i}^{t_{i-1}}\right]\phi_1(t)\,\frac{\sin \lambda_r t}{t}\,dt \\ &= I_1 + I_2 + I_3, \end{aligned}$$

say, and

$$I_1 = \int_0^{t_r} o\left(1/\log\log\frac{1}{t}\right)\frac{\sin \lambda_r t}{t} = \lambda_r \int_0^{\pi/\lambda_r} o(1)dt = o\,(1),$$

as $r \to \infty$.

$$\begin{aligned} I_2 &= c_r \int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} \frac{\sin^2 \lambda_r t}{t}\,dt = \frac{c_r}{2}\int_{\pi/\lambda_r}^{\pi/\lambda_{r-1}} \frac{1 - \cos 2\lambda_r t}{t}\,dt \\ &= \frac{c_r}{2}\log\frac{\lambda_r}{\lambda_{r-1}} - \frac{c_r}{2}\int_{2\pi}^{2\cdot 3^{2r-1}\pi} \frac{\cos u}{u}\,du \\ &= \frac{1}{2(2r-1)^{1/2}}\log 3^{2r-1} - \frac{1}{2}\cdot\frac{1}{(2r-1)^{1/2}}\,O(1) \\ &= \frac{(2r-1)^{1/2}}{2}\log 3 + o(1). \end{aligned}$$

Thus

$$I_2 \to \infty, \qquad \text{as } r \to \infty.$$

Again

$$
\begin{aligned}
I_3 &= \sum_{i=1}^{r-1} c_i \int_{t_i}^{t_{i-1}} \frac{\sin \lambda_i t \sin \lambda_r t}{t}\,dt \\
&= \sum_{i=1}^{r-1} \frac{c_i}{2} \int_{t_i}^{t_{i-1}} \frac{\cos(\lambda_r - \lambda_i)t - \cos(\lambda_r + \lambda_i)t}{t}\,dt \\
&= \sum_{i=1}^{r-1} \frac{c_i}{t_i} O\left(\frac{1}{\lambda_r - \lambda_i}\right) = \sum_{i=1}^{r-1} \frac{1}{(2i-1)^{1/2}} O\left(\frac{\lambda_i}{\lambda_r - \lambda_i}\right) \\
&= O\left\{\frac{r3^{(r-1)^2}}{3^{r^2} - 3^{(r-1)^2}}\right\} = O\left(\frac{r}{3^{2r-1} - 1}\right) = o(1),
\end{aligned}
$$

as $r \to \infty$.

Combining our results, we have

$$I_{\lambda_r} \to \infty, \qquad \text{as } r \to \infty.$$

References

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2. E. W. Hobson, *The theory of functions of a real variable*, vol. 2, 1926.

3. B. N. Prasad, *On the summability* (*C*, 1) *of Fourier series*, Math. Zeit. vol. 40 (1935) pp. 496–502.

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