A NOTE ON THE MEAN VALUE OF THE POISSON KERNEL

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In some investigations it is necessary to evaluate the mean value of some power of the Poisson kernel,

$$P(r, \theta) \equiv (1 - r^2)/(1 - 2r \cos \theta + r^2),$$

with respect to θ. This note gives a closed expression for this mean value, and an exact statement of the order of growth as r approaches 1.

THEOREM 1. *If* $x = 2r/(1+r^2)$, *then*

$$\frac{1}{2\pi}\int_0^{2\pi} P^{n+1}(r, \theta)d\theta = \left(\frac{1 - r^2}{1 + r^2}\right)^{n+1} \cdot \frac{1}{\Gamma(n + 1)} \cdot \frac{d^n}{dx^n}\left(\frac{x^n}{(1 - x^2)^{1/2}}\right), \qquad n > -1. \tag{1}$$

If n is not an integer the derivative is to be computed by the formula of Riemann and Liouville[1]

$$\frac{d^n}{dx^n}(f(x)) = \frac{d^m}{dx^m}\frac{1}{\Gamma(\rho)}\int_0^x (x - t)^{\rho-1}f(t)dt, \tag{2}$$

where m *is the smallest integer not less than* n *and* $\rho = m - n$.

The proof consists merely of the comparison of two power series. Clearly

$$P^{n+1}(r, \theta) = \left(\frac{1 - r^2}{1 + r^2}\right)^{n+1}\left(1 - \frac{2r}{1 + r^2}\cos \theta\right)^{-(n+1)},$$

and the second parenthesis, with $x = 2r/(1+r^2)$, is $1 + (n+1)x \cos \theta + (n+1)(n+2)/2!x^2 \cos^2 \theta + \cdots$ by the binomial theorem. Since

$$\begin{aligned}\int_0^{2\pi} \cos^p \theta d\theta &= 0 &&\text{(if } p \text{ is an odd integer)}\\ &= \frac{4(p - 1)(p - 3) \cdots 3\cdot 1}{p(p - 2) \cdots 4\cdot 2}\cdot\frac{\pi}{2} &&\text{(if } p \text{ is even)}\end{aligned}$$

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[1] See, for example, Courant, *Differential and integral calculus*, rev. ed., vol. 2, pp. 339–340.

equation (1) can be written

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} P^{n+1}(r,\theta)d\theta \\
&\quad = \left(\frac{1-r^2}{1+r^2}\right)^{n+1}\left[1+\frac{(n+1)(n+2)}{2!}\cdot x^2\cdot\frac{1}{2}\right. \\
&\qquad \left. + \frac{(n+1)(n+2)(n+3)(n+4)}{4!}\cdot x^4\cdot\frac{1\cdot 3}{2\cdot 4}+\cdots\right] \\
&\quad = \left(\frac{1-r^2}{1+r^2}\right)^{n+1}\left[1+\sum_{k=1}^{\infty}\frac{(n+1)(n+2)\cdots(n+2k)}{(2k)!}\right. \\
&\qquad \left. \cdot\frac{1\cdot 3\cdot 5\cdots(2k-1)}{2^k\cdot k!}\cdot x^{2k}\right].
\end{aligned}
\tag{3}
$$

Now

$$
\begin{aligned}
x^n(1-x^2)^{-1/2} &= x^n+\frac{1}{2}x^{n+2}+\frac{1\cdot 3}{2^2\cdot 2!}x^{n+4} \\
&\quad +\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^{n+6}+\cdots .
\end{aligned}
\tag{4}
$$

If n is an integer, this can be differentiated n times to yield

$$
\begin{aligned}
&n!+\frac{(n+2)!}{2!}\cdot\frac{1}{2}\cdot x^2+\frac{(n+4)!}{4!}\cdot\frac{1\cdot 3}{2^2\cdot 2!}x^4 \\
&\quad +\frac{(n+6)!}{6!}\cdot\frac{1\cdot 3\cdot 5}{2^3\cdot 3!}x^6+\cdots .
\end{aligned}
$$

Division by $\Gamma(n+1)$, as required in (1), produces the power series of (3).

If n is not an integer, formula (2) is applied to (4) to yield, on the right, a series of terms containing integrals of the form

$$
\frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}t^{n+2p}dt, \qquad p=0,1,2,\cdots .
$$

The substitution $t=xu$ changes these to

$$
\begin{aligned}
\frac{x^{n+2p+\rho}}{\Gamma(\rho)}\int_0^1 (1-u)^{\rho-1}u^{n+2p}du &= \frac{x^{m+2p}}{\Gamma(\rho)}\cdot B(\rho, n+1+2p) \\
&= \frac{x^{m+2p}\Gamma(n+2p+1)}{\Gamma(n+2p+1+\rho)},
\end{aligned}
$$

and the mth derivative of one of these terms is

$$\frac{(m+2p)!x^{2p}\Gamma(n+2p+1)}{(2p)!\Gamma(n+2p+1+\rho)}.$$

Hence the right member of (4) becomes

$$\frac{m!\Gamma(n+1)}{\Gamma(n+1+\rho)}+\frac{1}{2}\cdot\frac{(m+2)!\Gamma(n+3)}{2!\Gamma(n+3+\rho)}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}\cdot\frac{(m+4)!\Gamma(n+5)}{4!\Gamma(n+5+\rho)}x^4+\cdots.$$

Since $n+\rho=m$, this reduces to

$$\Gamma(n+1)\left[1+\frac{1}{2}\cdot\frac{(n+1)(n+2)}{2!}x^2 + \frac{1\cdot 3}{2^2\cdot 2!}\cdot\frac{(n+1)(n+2)(n+3)(n+4)}{4!}x^4+\cdots\right],$$

the desired series.

The integration term by term is justified, since $(x-t)^{\rho-1}$ is integrable and the series which it multiplies is uniformly convergent.

The order of growth of this mean value, as r approaches 1, is specified by the following theorem.

THEOREM 2.

$$\lim_{r\to 1-}\frac{(1-r)^n}{2\pi}\int_0^{2\pi}P^{n+1}(r,\theta)d\theta = \frac{(2m)!\Gamma(n+1/2)}{m!2^m\Gamma(n+1)\Gamma(n+1/2+\rho)}, \qquad n>-1/2, \tag{5}$$

where m, n and ρ are related as in Theorem 1.

If n is an integer the right member reduces to $(2n)!/(n!)^2 2^n$.

PROOF. Since $(1-r^2)^2/(1+r^2)^2=1-x^2$, where x has the same meaning as in Theorem 1, and since $(1+r)/(1+r^2)$ will approach unity, it is convenient to replace $(1-r)^n$ by $(1-x^2)^{n/2}$, and prove that the right member of (5) is equal to

$$\lim_{x\to 1-}(1-x^2)^{n/2}(1-x^2)^{(n+1)/2}\frac{1}{\Gamma(n+1)}\cdot\frac{d^n}{dx^n}(x^n(1-x^2)^{-1/2}). \tag{6}$$

If $n=0$, this is obviously true. If n is a positive integer, let

$\phi(x)=(1-x^2)^{-1/2}$, and consider the derivatives of $x^n\phi(x)$.

$$d(x^n\phi(x))/dx = x^{n+1}\phi^3(x) + nx^{n-1}\phi(x);$$
$$d^2(x^n\phi(x))/dx^2 = 3x^{n+2}\phi^5(x) + \text{terms in lower powers of } \phi(x);$$

and by mathematical induction it can readily be shown that

$$d^n(x^n\phi(x))/dx^n = 1\cdot 3\cdot 5 \cdots (2n-1)x^{2n}\phi^{2n+1}(x) + \text{terms in } \phi^{2n-1}, \phi^{2n-3}, \cdots, \phi.$$

Since

$$\lim_{x\to 1-} (1-x^2)^{n+1/2}\phi^p(x) \begin{cases} = 0, & p = 1, 3, 5, \cdots, 2n-1, \\ = 1, & p = 2n+1, \end{cases}$$

the limit in (6) is

$$\frac{1\cdot 3\cdot 5\cdots(2n-1)}{\Gamma(n+1)} = \frac{(2n)!}{2^n(n!)^2},$$

as required in the theorem.

If n is not an integer,[1]

$$\frac{d^n}{dx^n}(x^n\phi(x)) = \frac{d^m}{dx^m}\frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}t^n(1-t^2)^{-1/2}dt$$
$$= \frac{1}{\Gamma(\rho)}\int_0^x (x-t)^{\rho-1}\frac{d^m}{dt^m}(t^n\phi(t))dt.$$

As before, it is necessary to consider only the first term of the derivative, $1\cdot 3\cdot 5 \cdots (2m-1)x^{m+n}\phi^{2m+1}(x)$, since

$$(1-x^2)^{n+1/2}\int_0^x (x-t)^{\rho-1}t^{m+n-2}\phi^{2m-1}(t)dt$$
$$< (1-x^2)^{n+1/2}\int_0^x \frac{(x-t)^{\rho-1}dt}{(1-x^2)^{m-1/2}}$$
$$= (1-x^2)^{1-\rho}\int_0^x (x-t)^{\rho-1}dt,$$

and this approaches zero. Consequently it is necessary to consider

$$\lim_{x\to 1-}(1-x^2)^{n+1/2}\int_0^x (x-t)^{\rho-1}t^{m+n}\phi^{2m+1}(t)dt.$$

This limit, multiplied by

$$\frac{1\cdot 3\cdot 5\cdots(2m-1)}{\Gamma(\rho)\Gamma(n+1)}, \tag{7}$$

is the result sought.

The substitution $t=x-(1-x)u$ reduces the integral to

$$\frac{x^{2m-\rho}(1-x^2)^{n+1/2+\rho}}{(1+x)^\rho(1-x^2)^{m+1/2}} \cdot\int_0^{x/(1-x)} \frac{u^{\rho-1}(1-(1-x)u/x)^{2m-\rho}du}{(1+u)^{m+1/2}(1-(1-x)u/(1+x))^{m+1/2}}. \tag{8}$$

Since $n+\rho=m$, and x is to approach 1 later, the factor outside the integral sign will have the limit $2^{-\rho}$. To evaluate the integral, let a be a number between 0 and 1 (the way to choose a will become clear later; a will depend on n but not on x), and consider

$$\int_{ax/(1-x)}^{x/(1-x)} (\text{same integrand as in (8)})\ du. \tag{9}$$

In the interval of integration, $u\geqq ax/(1-x)$, $1+u\geqq(1-x(1-a))/(1-x)$, and $1-(1-x)u/(1+x)\geqq 1/(1+x)$. Hence the integral in (9) is less than

$$((1-x)/ax)^{1-\rho}(1+x)^{m+1/2}\{(1-x)/(1-x(1-a))\}^{m+1/2} \cdot\int_{ax/(1-x)}^{x/(1-x)}(1-(1-x)u/x)^{2m-\rho}du.$$

Except for a bounded factor this is $(1-x)^{m+3/2-\rho}x(1-a)^{2m+1-\rho}/(1-x)$, and accordingly approaches zero as x approaches 1; if n is negative, m is zero, and $n=-\rho>-1/2$.

Therefore the desired limit can be found by replacing the upper limit of integration in (8) by $ax/(1-x)$. This new integral will be not less than

$$\int_0^{ax/(1-x)} \frac{u^{\rho-1}(1-a)^{2m-\rho}}{(1+u)^{m+1/2}}\,du. \tag{10}$$

As x approaches 1 this has the limit $(1-a)^{2m-\rho}B(\rho,\ m-\rho+1/2)$. Also, the new integral will not be greater than

$$\int_0^{ax/(1-x)} \frac{u^{\rho-1}du}{(1+u)^{m+1/2}(1-ax/(1+x))^{m+1/2}} = \left(\frac{1+x}{1+x(1-a)}\right)^{m+1/2}\int_0^{ax/(1-x)}\frac{u^{\rho-1}du}{(1+u)^{m+1/2}}, \tag{11}$$

which has the limit $2^{m+1/2}/(2-a)^{m+1/2}B(\rho,\ m-\rho+1/2)$. If n is negative, m is zero, and the factor $(1-a)^{2m-\rho}$ will appear in (11) instead of (10).

Since a can be taken as close to zero as is desired, it follows that the limit exists and is $B(\rho,\ m-\rho+1/2)$, or $B(\rho,\ n+1/2)$. If the factor (7) is annexed, the theorem follows.

The theorem cannot be extended to the case $-1<n\leqq -1/2$, for if n is replaced by $-\rho$,

$$\frac{(1-r)^n}{2\pi}\int_0^{2\pi}P^{n+1}(r,\theta)d\theta = \frac{(1-r)^{-\rho}}{2\pi}\left(\frac{1-r^2}{1+r^2}\right)^{1-\rho}\int_0^{2\pi}\frac{d\theta}{(1-x\cos\theta)^{1-\rho}}\cdot$$

After a bounded factor is removed, the substitution $u=\cos\theta$ gives

$$(1-x)^{1/2-\rho}\int_{-1}^{+1}\frac{du}{(1-u^2)^{1/2}(1-xu)^{1-\rho}}\cdot$$

Since this integral converges at $u=-1$, it is necessary to consider only the interval from 0 to 1 to show divergence. The change of variable $xu=t(1-x)$ yields

$$(1-x)^{1/2-\rho}\int_0^{x/(1-x)}\frac{dt(1-x)/x}{(1-(1-x)t/x)^{1/2}(1+(1-x)t/x)^{1/2}(1-t(1-x))^{1-\rho}},$$

which is greater than

$$\frac{(1-x)^{3/2-\rho}}{2^{1/2}}\int_0^{x/(1-x)}(1-(1-x)t/x)^{-1/2}dt = (1-x)^{1/2-\rho}2^{1/2}x.$$

Hence, if ρ is greater than 1/2, the integral diverges and no limit exists.

If $n=-1/2$, the theorem fails to hold, but the order of growth of the mean value can be found. Here $m=0$, $\rho=1/2$, and

$$\frac{(1-r)^n}{2\pi}\int_0^{2\pi}P^{n+1}(r,\theta)d\theta = \frac{(1-r)^{-1/2}}{2\pi}\left(\frac{1-r^2}{1+r^2}\right)^{1-1/2}\int_0^{2\pi}(1-x\cos\theta)^{-1/2}d\theta.$$

The last integral can be written

$$\int_0^{2\pi}(1-x+2x\sin^2\theta/2)^{-1/2}d\theta,$$

which is greater than

$$\int_0^{2\pi} (1 - x + 2 \sin^2 \theta/2)^{-1/2} d\theta.$$

If $1-x$ is set equal to $2\alpha^2$, the last integral is greater than

$$\frac{1}{2^{1/2}} \int_0^{2\pi} (\alpha^2 + \theta^2)^{-1/2} d\theta = \frac{1}{2^{1/2}} \log (2\pi + (4\pi^2 + \alpha^2)^{1/2})/\alpha,$$

which becomes infinite as α approaches zero. Now $|\log \alpha|$ is effectively $|\log (1-x)|$ or $|\log (1-r)|$ multiplied by a constant. By an estimation similar to the foregoing, the integral can be shown to be less than $|\log (1-r)|$ multiplied by a second constant. Hence, if $n = -1/2$, the order of growth of the mean value is $(1-r)^{1/2} |\log(1-r)|$.

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