free. An elementary counting process used in the proof yields an asymptotic formula for a number $S(n)$ of square-free numbers of the form ax^2+b, $x \leqq n$. The method of proof can be used for several generalizations. (Received January 7, 1947.)

119. R. D. Wagner: *The generalized Laplace equations in a function theory for commutative algebras.*

The author considers the generalization of the theory associated with the Laplace equation $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$ in the theory of functions of a complex variable to that of functions in a commutative linear associative algebra with a principal unit. The usual theory extends readily if the algebra is a Frobenius algebra. Using matric methods introduced by Ward (Duke Math. J. vol. 7 (1940)), the author obtains the generalized Cauchy-Riemann equations as necessary and sufficient conditions for analyticity of a function in the algebra A of order n. The generalized Laplace equations are then obtained as necessary and sufficient conditions that a function u of n variables shall be a component of an analytic function. One form of the generalized Laplace equations is equivalent to the statement that the Hessian matrix of u shall be the parastrophic matrix of some number of A. (Received December 10, 1946.)

Analysis

120. Richard Bellman: *On the boundedness of solutions of nonlinear differential and difference equations.*

The purpose of the author is to discuss the behavior of solutions of the nonlinear system of differential equations: $dx_i/dt = \sum_{j=1}^{N} a_{ij}x_j + f_i(x_1, x_2, \cdots, x_N, t)$, $i = 1, 2, \cdots, n$, as $t \to \infty$, under various restrictions upon the matrix (a_{ij}), the functions $f_i(x, t)$, and the initial values. The more general case where the right-hand side contains the derivatives dx_i/dt is also considered. The three methods used are the method of successive approximations, the fixed-point method due to Birkhoff and Kellogg, and the method of approximating to a differential equation by a difference equation. Analogous results are derived for nonlinear difference equations. (Received January 22, 1947.)

121. Stefan Bergman: *Functions satisfying linear partial differential equations and their properties.*

The author investigates functions $\psi(\theta, \mathrm{H})$ of two real variables which satisfy the equation $(-\mathrm{H})^s\psi_{\theta\theta} + \psi_{\mathrm{HH}} = 0$ for $\mathrm{H} \leqq 0$, and $(-\mathrm{H})^s\psi_{\theta\theta} + \psi_{\mathrm{HH}} = 0$, for $\mathrm{H} \geqq 0$, $s > -2$, $\psi_{\theta\theta} \equiv (\partial^2\psi/\partial\theta^2)$, $\cdots$. In this paper the initial-value problem is considered. To this end $\psi(\theta, \mathrm{H})$ is expressed in terms of $T^{(0)}(\theta)$ and $T^{(1)}(\theta)$, the prescribed values of ψ and $\partial\psi/\partial\mathrm{H}$ respectively on the line $\mathrm{H} = 0$. It is shown that the function ψ satisfying the conditions $\psi(\theta, 0) = 0$, $\psi_{\mathrm{H}}(\theta, 0) \equiv [\partial(\theta, \mathrm{H})/\partial\mathrm{H}]_{\mathrm{H}=0} = T^{(1)}(\theta)$, $\theta^{(0)} \leqq \theta \leqq \theta^{(1)}$, where $T^{(1)}(\theta)$ is an analytic function of the real variable θ, can be written in the form $2\pi^2 i(2+s)\psi(\theta, \mathrm{H}) = \mathrm{H}\int_{\chi=0}^{1}\int_{t=0}^{\pi}\int_C (1-\chi)^{(s+2)^{-1}} T^{(1)}(\vartheta)\,dt\,d\chi\,d\vartheta/(\vartheta-\theta)[1+4(2+s)^{-2}(\vartheta-\theta)^{-2}(-\mathrm{H})^{s+2}\chi \sin^2 t]$, $\mathrm{H} < 0$, and a similar expression holds for $\mathrm{H} > 0$. C is a simple closed curve in the regularity domain of $T^{(1)}(\vartheta)$ (considered as a function of the complex variable $\vartheta = \theta + i\Theta$), which curve includes the interval $\theta^{(0)} \leqq \theta \leqq \theta^{(1)}$ of the real axis. An analogous formula holds for the solution ψ satisfying the condition $\psi(\theta, 0) = T^{(0)}(\theta)$, $\psi_{\mathrm{H}}(\theta, 0) = 0$. Using these formulae, the author investigates the connections which exist between the location and the nature of the singularities of $T^{(0)}(\vartheta)$ and

$T^{(1)}(\vartheta)$ on the one hand, and the properties (in the large) of $\psi(\theta, \mathrm{H})$ on the other. (Received January 30, 1947.)

122. Stefan Bergman: *On functions satisfying certain partial differential equations of elliptic type and their representation.* I.

The author considers function pairs (ϕ, ψ) satisfying the system of equations: $\phi_x = l\psi_y$, $\phi_y = -l\psi_x$, where $l = l(x, y)$ is a differentiable function which is positive in a domain $\mathfrak{P}$. Let $\mathfrak{B}$, $\overline{\mathfrak{B}} \subset \mathfrak{P}$, be a connected domain bounded by $(n+1)$ curves $\mathfrak{a}_0, \mathfrak{a}_1, \cdots, \mathfrak{a}_n$. The author defines single-valued functions ϕ and ψ which possess logarithmic singularities and singularities of nth order respectively at a prescribed point (x_0, y_0). In analogy to the case of analytic functions of a complex variable, he introduces integrals $(\phi^{(k,s)}, \psi^{(k,s)})$, $k = 1, 2, \cdots, n$, of the first kind and sth type, $s = 1, 2$. Here $\phi^{(k,1)}$ and $\psi^{(k,2)}$ are single-valued and regular in $\mathfrak{B}$, and are equal to 1 on $\mathfrak{a}_k$ and vanish on the remaining $\mathfrak{a}_s$. Integrals $(\phi^{(L,s)}, \psi^{(L,s)})$ of the third kind are then defined. $\phi^{(L,1)}$ and $\psi^{(L,2)}$ are single-valued functions and possess at the prescribed point $(x_0, y_0) \in \mathfrak{B}$ a logarithmic singularity. It is proved that $\phi^{(k,1)}(x_0, y_0) = (-1/2\pi) \oint_{\mathfrak{a}_k} \phi^{(L,1)}(x, y; x_0, y_0) ds$; various other relations between the periods of the above integrals are derived. (Received December 23, 1946.)

123. Stefan Bergman: *On functions satisfying certain partial differential equations of elliptic type and their representation.* II.

The author considers functions satisfying the equation $L(\phi) = \phi_{xx} + \phi_{yy} + 4P\phi = 0$, where $P = P(x, y)$ is nonpositive in the domain $\mathfrak{B}$ under consideration. Two solutions $\phi^{(\nu)}, \phi^{(\mu)}$ are said to be orthonormal in $\mathfrak{B}$ with respect to L if $\{\phi^{(\nu)}, \phi^{(\mu)}\}_{L\mathfrak{B}} = \iint_{\mathfrak{B}} [\phi_x^{(\nu)}\phi_x^{(\mu)} + \phi_y^{(\nu)}\phi_y^{(\mu)} - 4P\phi^{(\nu)}\phi^{(\mu)}] dx dy = \delta_{\nu\mu}$. ($\delta_{\nu\mu}$ is the Kronecker delta.) Let $(\phi^{(\nu)})$, $\nu = 1, 2, \cdots$, be a system of solutions of L which are orthonormal in $\mathfrak{B}$ with respect to L. It is proved that $\sum_{\nu=1}^{\infty} [\phi^{(\nu)}(x, y)]^2$, $\sum_{\nu=1}^{\infty} [\phi_x^{(\nu)}(x, y)]^2$, $\sum [\phi_y^{(\nu)}(x, y)]^2$ converge uniformly and absolutely in every closed subdomain which lies in $\mathfrak{B}$. If the system $(\phi^{(\nu)})$ is complete with respect to the class of solutions ϕ of L for which $\{\phi, \phi\}_{L\mathfrak{B}} < \infty$, then every solution ϕ can be represented in the form $\phi = \sum_{\nu=1}^{\infty} a_\nu \phi_\nu$, $a_\nu = \oint_{\mathfrak{c}} \phi (\partial \phi^{(\nu)} / \partial n) ds = \oint_{\mathfrak{c}} \phi^{(\nu)} (\partial \phi / \partial n) ds$. Here n is the inward normal and ds a line element of the boundary $\mathfrak{c}$ of $\mathfrak{B}$. (Received December 23, 1946.)

124. Stefan Bergman and Menahem Schiffer: *A representation of Green's and Neumann's functions in the theory of partial differential equations of second order.* Preliminary report.

Generalizing results by Schiffer (Duke Math. J. vol. 13 (1946) pp. 529–540) which permit the representation of Green's function of Laplace's equation in terms of orthonormal analytic functions of one complex variable, and using orthogonal functions introduced in the previous abstract, the authors consider the Green's function $G(Z; \zeta)$ and Neumann's function $N(Z; \zeta)$, $Z = (x, y)$, $\zeta = (\xi, \eta)$, of differential equations $L(\phi) \equiv \Delta\phi + 4P\phi = 0$, where $P \leqq 0$ in the domain under consideration. Let $\mathrm{K}(Z; \zeta) = \sum_{\nu=1}^{\infty} \phi^{(\nu)}(Z) \phi^{(\nu)}(\zeta)$ be the kernel function of a complete system of functions which are orthonormal in $\mathfrak{B}$ with respect to L, that is, for which $\{\phi^{(\nu)}, \phi^{(\mu)}\}_{L\mathfrak{B}} = \delta_{\nu\mu}$. (See the previous abstract.) Every solution ϕ of $L(\phi) = 0$ satisfies the integro-differential equation $\phi(Z) = \{\mathrm{K}(Z; \zeta), \phi(\zeta)\}_{L\mathfrak{B}}$. From this result it follows that $\mathrm{K}(Z; \zeta) = (2\pi)^{-1}(G(Z; \zeta) - N(Z; \zeta))$. Let $S(Z; \zeta)$ be a fundamental solution of $L(S) = 0$. Then $G(Z; \zeta) = S(Z; \zeta) + \{S(Z, T), \mathbf{K}(T; \zeta)\}_{L\mathfrak{B}}$. The variation $\delta\mathrm{K}(Z; \zeta)$, when $\mathfrak{B}$

varies, is given by $\oint_{\mathfrak{c}}[K(Z; T)K(T; \zeta)]\delta n ds$, $T=(u, v)$, $[\phi, \psi]=\phi_u\psi_u+\phi_v\psi_v-4P\phi\psi$, n being the inward normal, $\mathfrak{c}$ the boundary curve of $\mathfrak{B}$, ds a line element of $\mathfrak{c}$. The same formulae hold for G and N. Analogous results can be obtained for much more general equations in two or n, $n>2$, variables. (Received December 23, 1946.)

125. V. F. Cowling: *Some results for Dirichlet series.*

Let $f(z)=\sum_{n=1}^{\infty}a_ne^{-(\log n)z}$ with abscissa of convergence $\sigma_c<\infty$. Let $l-1<h<l$ where l is integral and positive but otherwise arbitrary. Denote by D the domain in the w-plane $\psi_2\leqq \text{Arg } (w-h)\leqq\psi_1$ where $0<\psi\leqq\pi/2$ and $-\pi/2\leqq\psi_2<0$. Let $a(w)$ be a function regular in D, with the possible exception of the point at infinity, for which $a(n)=a_n$, $n=0, 1, 2, \cdots$. Suppose, for $w=h+Re^{i\psi}$ in D and $R\geqq R_1$, that $|a(h+Re^{i\psi})|<R^K$ for some K. Then $f(z)=g(z)+h(z)$ where $g(z)$ is an integral function of z and $h(z)$ is regular for z in any closed bounded domain contained in $\mathfrak{R}(z)>K+1$. If for $w=h+Re^{i\psi}$ in D and $R\geqq R_1$, $|a(h+Re^{i\psi})|<R^Ke^{-LR\sin\psi}$ for some K and some $0<L<2\pi$, then $f(z)$ defines an integral function of z. (Received January 24, 1947.)

126. J. B. Díaz and H. J. Greenberg: *Upper and lower bounds for the solution of the general biharmonic boundary value problem.*

The authors consider the equation $\Delta\Delta w=p(x, y)$ in a domain D, subject to boundary conditions on w and $\partial w/\partial n$. Let U and V be the sets of all functions satisfying the boundary conditions and the differential equation, respectively. Integral expressions involving arbitrary functions of U and V are found which provide upper and lower bounds for the solution w at any given point of D. The method uses Fourier type expansions for w in functions of U and V separately to yield Bessel inequalities bounding an integral of w from above and below. (These inequalities are interpretable as complementary variational principles for w; one principle extending over the class U, the other over the class V.) Introduction of a boundary value problem, connected with the fundamental solution of the biharmonic equation, leads to bounds directly on w at the given point of D. The bounds are successively improved as additional Fourier coefficients are introduced. Conditions are established for convergence of the bounds to the desired value of w. (Received January 22, 1947.)

127. R. J. Duffin: *Function classes invariant under the Fourier transform.*

A function $f(x)$ belongs to class J if: (a) $(-xd/dx)^nf(x)\geqq 0$; $0<x<\infty$ $n=0, 1, 2, \cdots$, (b) $f(x)\to 0$ as $x\to\infty$, (c) $xf(x)\to 0$ as $x\to 0$. It is shown that the Fourier sine transform of $f(x)$ also belongs to the class J. For example, $x^{-1/3}$ is obviously of class J and its transform is $cx^{-2/3}$. A self-contained proof is given which hinges on a lemma which asserts that $x^{-1}f(x^{-1})$ is of class J. With a slight modification a similar theorem is valid for the cosine transform. Another invariant class K is characterized essentially by the condition $(-d/xdx)^nf(x)\geqq 0$. (Received December 23, 1946.)

128. J. J. Gergen and F. G. Dressel: *A minimal problem for harmonic functions.*

Let $\{\theta_k\}$, $\{h_k\}$, $k=1, 2, \cdots, n$, be arbitrary sets of n real numbers, $0\leqq\theta_1<\theta_2<\cdots<\theta_n<2\pi$. Let $\lambda_1(\theta)$, $\lambda_2(\theta)$ be step functions of period 2π determined by $\lambda_1(\theta)=\min(h_k, h_{k+1})$, $\lambda_2(\theta)=\max(h_k, h_{k+1})$, $\theta_k\leqq\theta<\theta_{k+1}$. Let Γ be the class of functions,

$U(r, \theta)$, each of which is harmonic in the interior S of the circle $r=1$ and satisfies $\lambda_1(\theta) \leqq \liminf U(r, \theta)$, $\limsup U(r, \theta) \leqq \lambda_2(\theta)$, $r \to 1-$, for almost all $0 \leqq \theta < 2\pi$. The authors consider minimizing in Γ the functional $A[U] = \iint_S |\nabla U|^2 dS$, where $|\nabla U|$ is the magnitude of the gradient of U. The chief results are: (1) The lower bound A_0 of $A[U]$ for $U \subset \Gamma$ is finite. (2) $A[U] = A_0$ at least once in Γ. (3) $A[U] = A_0$ at most once in Γ unless all the h_k of even index are greater than, or are less than, all those of odd index. (4) If $U \in \Gamma$, $A[U] = A_0$, and if $F(z)$ is analytic for $|z| < 1$ and has U as real part, then $[zF'(z)]^2 = i\sum_{k=1}^{n} B_k(z+z_k)/(z-z_k)$, $z_k = e^{i\theta_k}$, where B_k are real constants such that $\sum_{k=1}^{n} B_k = \sum_{k=1}^{n} B_k/z_k = 0$. For $n=4$, A_0 and F are determined in terms of θ_i and h_i. The analysis is based on Douglas' paper (Trans. Amer. Math. Soc. vol. 33 (1931) pp. 263–321). (Received December 12, 1946.)

129. Irving Gerst: *Meromorphic functions with simultaneous multiplication and addition theorems.*

This paper is devoted to the characterization of all meromorphic functions $f(z)$ for which there exist simultaneously relations of the form $f(mz) = R[f(z)]$, $f(z+h) = S[f(z)]$, where m and h are complex numbers and R and S are rational functions. When $|m| > 1$, the case of chief interest, it is shown that if $f(z)$ is transcendental it must be periodic. Then, if results of Ritt (Trans. Amer. Math. Soc. vol. 23 (1922) pp. 16–25) on periodic functions with multiplication theorems are utilized, it is found that except for $f(z)$ a linear function of $e^{\alpha z}$, h, when not a period, must be a suitable half or third of a period. The corresponding functions and their addition theorems are given explicitly. Other results are: (1) If $f(z)$ is rational it must be a linear function of z. (2) The linear function of z is the only type possible when $|m| \leqq 1$ except when m is a rational root of unity. (Received January 11, 1947.)

130. R. N. Haskell and W. H. Bradford: *Sub-biharmonic functions.* Preliminary report.

A definition of this class of functions is made by use of areal derivatives as defined by Ridder (Nieuw Archief voor Wiskunde (2) vol. 16 (1929–1930) p. 63). Let $D_{x,y}\theta_B(J)$ denote the areal derivative of a biharmonic function $B(x, y)$ and let $u(x, y)$ be of class c'. If $D_{x,y}\theta_u(J)$ is subharmonic in a domain G and if $D_{x,y}\theta_u(J) \leqq D_{x,y}\theta_B(J)$ and also $u(x, y) \leqq B(x, y)$ on boundary of any subdomain G' of G, then $u(x, y) \leqq B(x, y)$ in interior of G' and $u(x, y)$ is sub-biharmonic. A characterization of sub-biharmonic functions is made in terms of areal and peripheral means. Let $L(u; x_0, y_0; r)$ and $A(u; x_0, y_0; r)$ represent the ordinary peripheral and areal means respectively on a circle with center at (x_0, y_0) and radius r. Further let L_1 and A_1 be defined by the relations $2\pi(1-\mu^2)L_1(u; x_0, y_0; \mu r, r) \leqq \int_0^{2\pi}[u(\mu r, \theta) - \mu^2 u(r, \theta)]d\theta$ and $\pi r^2(1-\mu^2)A_1(u; x_0, y_0; \mu r, r) \leqq \int_0^r\int_0^{2\pi}[u(\mu s, \theta) - \mu^2 u(s, \theta)]s\,ds\,d\theta$ where $0 < \mu < 1$ but fixed. It is shown that $L_1 \geqq A_1$ and both are increasing functions of r. Necessary and sufficient conditions that $u(x, y)$ shall be sub-biharmonic in G are that for all circles in G either $u(x_0, y_0) \leqq 2A - L$, $u(x_0, y_0) \leqq L_1$, or $u(x_0, y_0) \leqq A_1$. (Received December 26, 1946.)

131. Tibor Rado: *On two-dimensional concepts of bounded variation and absolute continuity.*

Let $x = x(u, v)$, $y = y(u, v)$ be continuous functions in the unit square Q: $0 \leqq u \leqq 1$, $0 \leqq v \leqq 1$. Then these equations define a continuous mapping T. For such a mapping, the author defined an essential multiplicity function $\mu(x, y)$ (See Duke Math. J. vol. 4 (1938) pp. 189–221). During the war, analogous researches were carried on in Italy by L. Cesari, whose work is based upon a certain multiplicity function $\Psi(x, y)$ termed the characteristic function of the mapping T (see Atti R. Accademia d'Italia vol. 13 (1943) pp. 1323–1481). The main purpose of the present paper is to show that $\Psi(x, y) = \mu(x, y)$ with the exception of a countable set. As a consequence, it follows that the concepts of bounded variation and of absolute continuity developed by Cesari for continuous mappings T agree with concepts introduced and studied by the author and by P. V. Reichelderfer (see Trans. Amer. Math. Soc. vol. 49 (1941) pp. 258–307 and vol. 53 (1943) pp. 251–291). (Received January 1, 1947.)

132. L. B. Robinson: *Generation of an infinite system of functions with the unit circle as singular line from the Garvin series.*

Let $G(x)$ be a Garvin series. Let $u(x) = \int_0^x G(x_1)u(x_1^p)dx_1 + u_0$ whose adjoint is $u(x) = \int_\infty^x G(x_1)u(x_1^p)dx_1 + v_0$. A solution $u_1(x)$ of the above has the unit circle as singular line like the Garvin series. In $u(x)$ replace $G(x)$ by $u_1(x)$. The solution $u_2(x)$ has the unit circle as singular line. Continue in this way until from the Garvin series is generated an infinite system of functions $u_1(x), u_2(x), \cdots, u_n(x), \cdots$ all of which admit the unit circle as singular line. (Received December 2, 1946.)

133. H. E. Salzer: *The approximation of numbers as sums of reciprocals.*

Every number x, $0 < x < 1$, admits a unique expansion of the form $x = (1/a_1) + (1/a_2) + \cdots$, where a_i are integers so chosen that after i terms, when the sum x_i has been obtained, a_{i+1} is the least integer such that $x_i + 1/a_{i+1}$ does not exceed x (denoted as an R-expansion). Similarly, to x_i one might add either $1/a_{i+1}$ or $1/(a_{i+1} - 1)$, depending upon which addend gives a small value to $|x - x_{i+1}|$ (denoted as an $\overline{R}$-expansion). It is shown that rational numbers have finite R- and $\overline{R}$-expansions. Properties investigated include rapidity of convergence, including the closeness of approximation of x_i when written as p_i/q_i, and necessary and sufficient conditions for a given sum of reciprocals to be either an R- or $\overline{R}$-expansion. These properties are applied to prove the irrationality of certain infinite sums S, where the criterion of $|S - p/q| < 1/q^2$ for an infinitude of integers p and q is not applicable. Both the R- and $\overline{R}$-expansions are applied to the solution of algebraic and transcendental equations by a well known variant of the Newton-Raphson method, with the amount of digital work considerably less than in the usual procedures. (Received December 4, 1946.)

134. I. E. Segal: *Postulates for general quantum mechanics.*

A set of postulates for the observables in a physical system is given. On the basis of these it is shown that for any two observables there is a pure state (in essentially the sense of von Neumann and Weyl) of the system in which they have different expectation values, a spectral resolution for each observable is obtained, probability distributions of simultaneously measurable observables are defined, and an indeterminacy principle is derived. The postulates are partly algebraic (the observables form a real linear space in which each element generates an associative algebra) and partly metric (each observable has a maximum numerical value which is a Banach norm and has certain special properties). Mathematical examples of systems satisfying the postulates are the self-adjoint elements of a uniformly closed self-adjoint operator-algebra, with the bound as norm, and the real-valued continuous functions on a compact Hausdorff space, with the maximum as norm. (Received January 6, 1947.)

135. A. E. Taylor: *The inverse of a polynomial function of a closed operator.*

Let T be a closed distributive operator with domain D and range both in the complex Banach space X. Further let the resolvent set R of T be nonempty. If $n \geqq 1$, let D_n be the set of elements x of X such that $x, Tx, \cdots, T^{n-1}x$ lie in D. The author considers the problem: given a polynomial $P(\lambda)$ of degree n, and an element y in X, to find an element x in D_n such that $P(T)x = y$. Under the assumption that all the zeros of $P(\lambda)$ lie in R the author proves that the solution x exists and is unique. It is given by $x = By$, where B is a linear operator with domain X and range D_n, defined by $2\pi i B = \int \{1/P(\lambda)\}(T-\lambda I)^{-1} d\lambda$, the integration being extended counterclockwise over a set of nonoverlapping circles, one each about the zeros of $P(\lambda)$, each circle and its interior lying in R. Thus, if the inverse operator $(T-\lambda I)^{-1}$ is known explicitly, B may be calculated by the calculus of residues. The proof employs spectral methods developed previously by the author for the case in which T is assumed to be continuous, with domain X. (Received January 30, 1947.)

136. Y. W. Tschen: *Existence of minimal surfaces with a simple pole and normal derivative equal to 0 on boundary.* Preliminary report.

The purpose of the author is to solve the problem: given a simple closed curve C in the xy-plane to find a minimal surface $z(x, y)$ with $z = (\text{const.}) \cdot x + \cdots$ at infinity and $\partial z/\partial n = 0$ on C. In parametric form the problem is: to find three harmonic functions $x'(u, v)$, $y'(u, v)$, $z'(u, v)$ defined in the interior of the unit circle $u^2+v^2<1$, which represent a minimal surface with a simple pole at (0, 0) and x', y' representing C, $\partial z'/\partial r = 0$ on $u^2+v^2=1$. This problem in the parametric form is solved by using direct methods of calculus of variations on the minimum problem for two functions. Conditions are studied under which x', y' are differentiable on the boundary and $\partial(x',y')/\partial(u, v) \neq 0$ in the interior. As soon as these two conditions are fulfilled, the function $z(x, y)$ is obtained. (Received December 13, 1946.)

137. D. W. Western: *Inequalities of the Markoff and Bernstein type for integral norms.*

Let $P(z)$ be a polynomial of degree n; A, constants independent of n and $P(z)$; C, a simple closed Jordan curve of length $L(C)$. Denote the unit circle by Γ, its closed interior by $\bar{\Gamma}$, the points of C satisfying $|z-k| > m$ for k on C by C^m. The curvilinear norm is defined by $\|P(z)\|_{p,C} = \{(1/L(C))\int_C |P(z)|^p |dz|\}^{1/p}$. The results follow. For smooth C, $\|P'(z)\|_{p,C} \leqq An\|P(z)\|_{p,C}$ for $p \geqq 1$. There is a constant B depending only on $p \geqq 1$ and C such that $\|F'(z)\|_{p,C} \geqq Bn\|F(z)\|_{p,C}$ for $F(z)$ the Faber polynomial of degree n associated with C, C analytic. If C consists of smooth arcs meeting in a finite number of corners of exterior angles $u_i\pi$ where $0 < u_i \leqq u < 2$, $u \geqq 1$, then $\|P'(z)\|_{p,C} \leqq An^{2u-1}\|P(z)\|_{p,C}$ for $p \geqq 1$; and $\|P'(z)\|_{p,C} \leqq An^{u(1+1/p)}\|P(z)\|_{p,C}$ for $p > 0$. If $P^{(\alpha)}(z)$ denotes the generalized derivative of the Riemann-Liouville definition then for $\alpha > 0$ and $p \geqq 1$, $\|P^{(\alpha)}(z)\|_{p,\Gamma^m} \leqq An^{\alpha}\|P(z)\|_{p,\Gamma}$ where m may satisfy $m = 1/n$ and $k = -1$; and $\|P^{(\alpha)}(x)\|_{p,(-1+m,1)} \leqq An^{2\alpha}\|P(x)\|_{p,(-1,1)}$ for $p > 1$, $0 < m < 1$; and $\|P^{(\alpha)}(z)\|_{p,C^m} \leqq An^{\alpha}\|P(z)\|_{p,C}$ for $p > 1$ and smooth C. Results of a similar nature are obtained in particular cases for area norms correspondingly defined, for $p = 2$, for certain functions not polynomials and for homogeneous polynomials of degree n in two real variables. (Received December 24, 1946.)

138. Bertram Yood: *On ideals in operator rings over Banach spaces.* Preliminary report.

Let R be the ring of all continuous linear operators on a Banach space X. For any ideal A in R let $M\subset X$ be the set of all x in X for which $T(x)=0$ for every T in A, and let N be the linear manifold in X generated by the ranges of T in A. It is shown that if A is a left (right) ideal in R then the closure of A in both the weak and strong topology of operators is the set of all T in R which vanish on all of M (set of all T in R whose range is contained in $\overline{N}$). Thus for ideals the topologies are equivalent and the closed ideals characterized. These facts are used to show that if S is a subset of R, the left (right) annihilator of the right (left) annihilator of S in R is the smallest weakly closed left (right) ideal containing S. (Received January 6, 1947.)

139. Bertram Yood: *Regular and singular elements in the ring of operators on a Banach space.* Preliminary report.

The sets G and S of regular and singular elements respectively of the ring R of all bounded linear operators defined on an infinite-dimensional Banach space X are studied in the uniform, strong and weak topologies of R. In the uniform topology, S is decomposed into disjoint sets with topological properties in such a way that the properties of an operator T being an isomorphism or not, having X as its range or not, and the existence or absence of certain projections are correlated with the ring properties of being a left or right generalized null-divisor or not, and the possession or lack of a left or right inverse. If G can be dense in R then an open question of Banach (*Théorie des opérations linéaires*, p. 246) on linear dimensions has a negative answer. Banach's question is further discussed in terms of the concepts of this paper. In both the strong and weak topologies of R, it is shown that G and S are each dense and not open. (Received January 31, 1947.)

140. H. J. Zimmerberg: *Definite integral systems.*

In this paper the notions of definitely self-conjugate adjoint integral systems of Wilkins (Duke Math. J. vol. 11 (1944) pp. 155–166) and those definite systems of the author (Bull. Amer. Math. Soc. Abstract 52-3-76) are extended to integral systems written in matrix form $y(x)=\lambda\int_a^b K(x, t)y(t)dt$, where no restriction is made on the form of the kernel $K(x, t)$. These definite integral systems include the definite systems previously treated as subclasses. With the exception of expansion theorems, most of the properties of the definite integral systems of Wilkins and the author are preserved. (Received January 8, 1947.)

Applied Mathematics

141. J. B. Díaz and Alexander Weinstein: *On an extremal property of the torsional rigidity.*

The torsional rigidity S of a beam with simply or multiply connected cross section can be given by either of the following formulas, which have been hardly explicitly mentioned in the literature: (*) $S=P-D(\phi)$, (**) $S=P-D(\psi)$, where P is the polar moment of inertia, ϕ and ψ are the warping function and its conjugate, and D denotes