BOOK REVIEWS

Les systèmes différentiels extérieurs et leurs applications géométriques. (Actualités Scientifiques et Industrielles, no. 994.) By Elie Cartan. Paris, Hermann, 1945. 214 pp. 450 fr.

This book gives a revised account of lectures delivered in 1936–1937 at the Sorbonne. The content is based almost exclusively on the author's own outstanding contributions to the subject at the beginning of the century. In developing many of these old results, however, the author's present viewpoint is new as well as stimulating. The whole treatment, even at the few points where it touches upon contemporary work by other writers, bears the stamp of the author's individuality.

The fundamental calculus employed is Grassmann algebra. Cartan's manner of regarding this discipline might be described as follows. The indeterminates are differentials and the polynomials are forms. Interest centers in skew-symmetric (necessarily multilinear) forms, which can be written

$$F = a_{i_1 \cdots i_p} \overset{i_1}{u_1} \cdots \overset{i_p}{u_p}, \tag{1}$$

where the coefficients a are skew-symmetric in every pair of indices and where the summation convention is being used. An equivalent expression for F is

$$F = \frac{1}{p!} a_{i_1 \cdots i_p} \begin{vmatrix} \overset{i_1}{u_1} & \cdots & \overset{i_p}{u_1} \\ \cdot & \cdot & \cdot \\ \overset{i_1}{u_p} & \cdots & \overset{i_p}{u_p} \end{vmatrix},$$

where the a's are the same as in (1). The exterior product of F by the form $G = b_{i_{p+1} \cdots i_{p+q}} \overset{i_{p+1}}{u_{p+1}} \cdots \overset{i_{p+q}}{u_{p+q}}$ is the skew-symmetric form defined by

$$[FG] = c_{i_1 \cdots i_{p+q}} \overset{i_1}{u_1} \cdots \overset{i_{p+q}}{u_{p+q}},$$

where the c's are found by applying all $(p+q)!$ signed permutations to the subscripts on $a_{i_1 \cdots i_p} b_{i_{p+1} \cdots i_{p+q}}$ and adding. The definition of product applied to $\overset{i_1}{u_1}, \cdots, \overset{i_p}{u_p}$ gives

$$[\overset{i_1}{u_1} \cdots \overset{i_p}{u_p}] = \begin{vmatrix} \overset{i_1}{u_1} & \cdots & \overset{i_p}{u_1} \\ \cdot & \cdot & \cdot \\ \overset{i_1}{u_p} & \cdots & \overset{i_p}{u_p} \end{vmatrix},$$

so that F may be written

$$F = \frac{1}{p!} a_{i_1 \cdots i_p} [u_1^{i_1} \cdots u_p^{i_p}],$$

or, if the subscripts on the u's are omitted, as

$$F = \frac{1}{p!} a_{i_1 \cdots i_p} [u^{i_1} \cdots u^{i_p}] \tag{2}$$

with the understanding that the interchange of two u's is to be compensated by multiplying by -1. When a skew-symmetric form is written in the symbolic notation (2), it is called an *exterior form.*

In this way, Cartan is able to construct simultaneously for his symbols two types of product, both of which can be evaluated for real values of the indeterminates. To distinguish between them a notation such as the brackets is necessary. As in the case of the vector product the use of brackets has its disadvantages: in the latter half of the book brackets are frequently used also as parentheses. Other writers, for example, Goursat in his *Problème de Pfaff*, regard the matter differently. One set of indeterminates (say, derivatives) is subjected solely to ordinary multiplication and another (say, differentials) solely to exterior multiplication. This not only makes the brackets unambiguously available for their normal use as fences but has manipulative advantages as well.

Besides developing Grassmann algebra along lines familiar to readers of Cartan's *Invariants intégraux*, Chapter I has several noteworthy features, which we proceed to mention.

On page 10 there is a theorem neatly contrasting an ordinary quadratic form F with an exterior quadratic form G, namely,

$$u^i \frac{\partial F}{\partial u^i} = 2F, \qquad u^i \frac{\partial G}{\partial u^i} = 0,$$

$$\left[u^i \frac{\partial F}{\partial u^i}\right] = 0, \qquad \left[u^i \frac{\partial G}{\partial u^i}\right] = 2G.$$

The theorem on linear dependence (p. 11) is also cast in this contrasting form.

The condition that a form be monomial (that is, have independent linear factors equal in number to its degree) is discussed at length. One might be tempted to dismiss the matter after saying that the associated system must have rank equal to the degree of the form. Cartan, however, finds in simple and elegant manner polynomials whose vanishing expresses the condition. It is interesting that the polynomials are quadratic. In the case of a quadratic exterior form, the facts could also be inferred from two familiar results: (i) a skew-symmetric matrix has rank at most two if and only if all principal determinants of order four are zero and (ii) a skew-symmetric determinant of order four is the square of a quadratic in its elements. If the theorem on matrices corresponding to Cartan's general result has been stated, it is at least not well known.

Instead of defining k-planes, as formerly, by the components of k independent vectors Cartan now employs the analogue of Plücker's line coördinates. At least as far as the applications in the present book are concerned, the increase in the number of coördinates and the necessity of satisfying the relations among them more than offset the advantages of their introduction. For example, brackets being omitted, the system (p. 27) $u^1u^3 = u^1u^4 = u^1u^2 - u^3u^4 = 0$ is at once seen to imply $u^1 = 0$ or $u^3 = au^1$, $u^4 = bu^1$, $u^3u^4 = 0$, $u^1u^2 = 0$ without appealing to Plücker coördinates and the quadratic relation among them. Conceivably, these coördinates may be elegant means for further developments in the theory. An extended treatment of their properties has been made by J. W. Givens, *Tensor coördinates of linear spaces*, Ann. of Math. (2) vol. 38 (1937) pp. 355–385.

The definition (p. 30) of the associated system for a system of equations is a noteworthy accomplishment, although it is not the ultimate because the rank gives the minimum number of indeterminates in an algebraically (defined on p. 28) equivalent system rather than simply in an equivalent system. The author remarks (p. 56) that the characteristic systems (which are associated systems) for two equivalent differential systems are not necessarily equivalent.

Chapter II discusses the differential dF of an exterior form F, its geometrical significance, and its connection with Stokes' theorem.

Chapters III, IV, V and VI give a systematic account of the author's theory of Pfaffian systems. Although most of the exposition follows established lines, it is decidedly advantageous to have this material united in one volume, particularly for the purpose of the geometrical applications in the last 90 pages.

The gist of this theory is as follows. The system considered consists of a finite number of exterior forms F_β in the differentials of n variables x^i with $0 \leqq \deg F_\beta \leqq n$. A solution is a set of functions $f_\alpha(x^1, \cdots, x^n)$ such that the vanishing of f_α, df_α implies the vanishing of F_β. An immediate consequence of the definition is that the system F_β is equivalent to the system F_β, dF_β, which is called *closed*. Attention accordingly can be confined to closed systems. From this standpoint

f_α constitutes a solution of F_β if and only if the closed system f_α, df_α implies the closed system F_β, dF_β.

Of fundamental importance are the linear Pfaffian system, for which each F_β is linear, and the completely integrable type of such a linear system, which is equivalent to df_β. The properties of these systems are developed in Chapter III. The condition for a completely integrable linear system is stated (p. 49) in terms of the "anneau" defined on p. 26; unfortunately the author has not followed Ore's lead (*L'Algèbre abstraite*, p. 6) in reserving "anneau" as a translation for "ring." Mayer's method (integration by means of ordinary equations) is treated in §37; its generalization (integration by means of equations in $k-1$ fewer independent variables) is indicated later (p. 78) for systems having k characters equal to zero. In reading §37 it may be helpful to insert the stipulation "of $n-r$ dimensions" into the statement of the theorem, to supply the arguments $z^1, \cdots, z^r$ on the right of (2), and to by-pass the question of the hypotheses on which the discussion rests by taking refuge in the general hypothesis of real, analytic functions (p. 45) and in Cauchy's theorem for ordinary differential equations. Later (p. 50, Remark IV) the assumption of the existence of first derivatives shows that (2) satisfies the Lipschitz condition even when its right members contain the z's. The well known integration of a single partial differential equation is expounded as an application of the theory of completely integrable systems.

Cartan's existence theorem, which is the backbone of the subject, involves the notion of *k-tangent* (or integral k-plane). If the form F_β has degree p, let its indeterminates be $u_1^{i_1}, \cdots, u_p^{i_p}$. A k-plane is defined by k independent vectors v_λ^i, $\lambda=1, \cdots, k$. This k-plane is a k-tangent for the system if and only if the set of forms F_α vanishes whenever each indeterminate vector u_α is identified with an arbitrary one of the vectors v_λ. If we start with a k-tangent and seek a $(k+1)$-tangent containing it, the system Σ_{k+1} upon the components of the $(k+1)$th vector v_{k+1}^i is linear and homogeneous. A point x^i is an *ordinary* 0-tangent if the forms of degree zero in F_β vanish at x^i and if the matrix of the linear equations in F_β has maximum rank at x^i. By induction, a k-tangent is defined as ordinary if it contains an ordinary $(k-1)$-tangent and if for it the system Σ_{k+1} has maximum rank. The notation E_k will be reserved for an ordinary k-tangent.

An integral variety of k dimensions is an *ordinary solution* if the k vectors tangent to its parametric curves constitute an E_k for every point. The notation V_k will be reserved for an ordinary solution of k dimensions.

The maximum dimension g for an ordinary k-tangent is called the *genus*. Cartan previously gave for the genus of a linear system a lower bound which is not repeated or generalized here. It is

$$(n - r)/(r + 1) \leqq g,$$

where n is the number of variables and r is the number of equations.

The existence theorem asserts there is a V_k if and only if k does not exceed the genus. More precisely, it states that if $k<g$, through V_k there passes a V_{k+1} depending on specified arbitrary elements. This is proved by an appeal to the Cauchy-Kowalevsky theorem.

A system is in involution with respect to a set of k variables $x^1, \cdots, x^k$ if it possesses a V_k on which $x^1, \cdots, x^k$ are independent, that is, on which $[dx^1 \cdots dx^k] \neq 0$. Necessary and sufficient conditions for a system to be in involution are given in terms of the characters, which are non-negative integers determined by the ranks of the systems Σ_i used to define E_k.

The final chapter on existence theorems describes the author's method of *prolongation*, whose purpose (p. 112) is to make each solution of a given system appear as an ordinary solution of a system in involution. As the author states (p. 120), singular solutions escape the theoretical discussion of prolongation, but for particular systems prolongation can be effected without this drawback. It is therefore recognized that the general process of prolongation and the field of its applicability have still to be rendered precise.

There are two fundamental requirements for a theory of differential systems: (1) a normal form for which a precise theorem can be rigorously proved; (2) a process for reduction to normal form. The process in (2) has two aspects: (2.1) formation of integrability conditions; (2.2) application of an implicit function theorem. Cartan's normal form is essentially a set of Cauchy-Kowalevsky systems and is less inclusive than Riquier's orthonomic system. Cartan's operation (2.1) involves prolongation, whereas Riquier's is precise. As for (2.2), the theories are subject to the same difficulties, but for algebraic systems these have been removed from Riquier's theory by Ritt.

The last two chapters give applications to the differential geometry of surfaces (Chapter VII) and of certain higher spaces (Chapter VIII). In addition to furnishing excellent illustrations for the preceding theory, they bring out in novel fashion the dependence of the solutions of the problems on the arbitrary elements. Conformal and isometric correspondences, Weingarten surfaces, isothermic surfaces and triply orthogonal systems are among the score of topics treated.

Schlaefli's theorem that every Riemannian space of n dimensions can be immersed in Euclidean space of $n(n+1)/2$ dimensions is discussed in great detail (pp. 199–210) for the particular case $n=3$ and reference is made to the proofs of Janet and Cartan for the general n. In this connection, it might be remarked that the detailed discussion given by Janet (Annales de la Société Polonaise de Mathématique vol. 5 (1926) pp. 39–40) for the case $n=2$ is convincing, whereas the counter-example given for the same case by Forsyth (*Intrinsic geometry of ideal space*, vol. 1, pp. 231–233) is not. A relatively simple proof of the theorem would be highly desirable. Cartan's discussion of the case $n=3$ may help in that direction.

J. M. Thomas

Tables of fractional powers. Prepared by Mathematical Tables Project, National Bureau of Standards. New York, Columbia University Press, 1946. 489+30 pp. $7.50.

The tables here printed yield the values of A^x and x^a. For example, there are tables of A^x for $A=10, \pi, 10^{-3}P$ (where P is a prime between 100 and 1000), as well as for other values. Thus 10^x is given to 15 decimals for $0.001 \leqq x \leqq 1.000$ with x advancing in intervals of .001. The function x^a is computed for the values $a = \pm 1/2, \pm 1/3, \pm 2/3, \pm 1/4$ with $0 \leqq x \leqq 9.99$ in intervals of .01. There is a bibliography with 76 titles and an introduction by Dr. Lowan in which the method of computation of the tables is explained and the accuracy of interpolation is illustrated by examples.

E. R. Lorch

Tables of the modified Hankel functions of order one-third and of their derivatives. Cambridge, Harvard University Press; London, Oxford University Press, 1945. 36+235 pp. $10.00.

This set of tables is the first to be published by the Computation Laboratory of Harvard University. The functions here considered are solutions of Stokes' differential equation $d^2u/dz^2+zu=0$ and were needed in connection with the work of the Radiation Laboratory on diffraction and refraction of waves. Solutions to Stokes' equation are $h_1(z)=(k/i\pi)\int_{L_1}e^{zt+t^3/3}dt$ (where k is a constant and L_1 is an infinite broken line in the complex plane) and $h_2(z)$, which has a similar expression. It is the functions $h_i(z)$ and their derivatives $h_i'(z)$ which are tabulated. The tables give the real and imaginary parts to eight decimal places for $z=x+iy$ with $|x+iy| \leqq 6$ and x, y progressing in intervals of 0.1. The functions $h_i(z)$ are related to the Hankel functions of order 1/3 by the equations $h_i(z)=((2/3)z^{3/2})^{1/3}H_{1/3}^{(i)}((2/3)z^{3/2})$,