GRATINGS AND HOMOLOGY THEORY

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1. **Introduction.** The relation of an abelian group to its character group is merely one example of a relation between an algebraic system Γ and a topological system X, each invariantively associated with the other. Of the two systems Γ and X, the first can be described in pseudo-combinatory terms, while the second involves the use of more sophisticated notions, such as passage to the limit, geometrical continuity, and so on. Accordingly, problems which are ordinarily expressed in terms of the system X can often be treated more simply by restating them in terms of the system Γ.

In this paper we shall be dealing with a class of algebraic systems Γ, called *gratings*. The theory of gratings will aim, among other things, to describe the topological properties of a space X in terms of the ways in which the space can be expressed as the union of two subspaces A and C, $A \cup C = X$. The theory will be pseudo-combinatory in character, in the sense that it will have to do with combinatory operations applied to an unlimited number of abstract elements, or symbols. It will acquire a geometrical significance only when the symbols are attached to appropriate geometrical entities. The theory will be applicable both to ordinary topological spaces and to spaces of a more general type, such as the ones determined by distributive lattices, which last need not be assumed to possess atomic elements. With the aid of gratings, we shall be able to reformulate a variety of problems in topology, differential geometry, potential theory, and so on, in pseudo-combinatory terms.

The present paper will be devoted almost exclusively to the elementary formal theory of gratings. Further developments, along with a number of typical applications, will be considered subsequently.[1]

2. **Cuts, elements, gratings, cells.** A *cut* will be any ordered triad $\gamma = (a, b, c)$ consisting of three different *abstract elements* a, b, and c. The element a will be called the *negative face*, the element b the *edge*, the element c the *positive face* of the cut.

A *grating* $\Gamma = [\gamma]$ will be any (finite or infinite) set of cuts, such that no two of the cuts have an element in common. The cuts γ will be

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[1] The terminology adopted in this paper differs quite radically from that used by the author in an earlier paper on gratings: *A theory of connectivity in terms of gratings*, Ann. of Math. vol. 39 (1938) pp. 883–912.

called the *cuts* of the grating, their elements will be called the *elements* of the grating. Given a cut $\gamma_i = (a_i, b_i, c_i)$, it will be convenient to introduce a symbol z_i to designate an unspecified element of γ_i, $z_i = a_i$, b_i, or c_i.

A *cell* A of a grating Γ will be any finite sequence of elements of Γ,

$$(2.1) \qquad A = z_1, z_2, \cdots, z_m \qquad (z_i = a_i, b_i, \text{ or } c_i).$$

The *degree* $m(A)$ of the cell will be the number of terms of the sequence, the *rank* $\rho(A)$ the number of terms in edges b_i, rather than in faces a_i or c_i. Thus we shall always have $0 \leqq \rho(A) \leqq m(A)$. A cell of rank ρ will be called a *ρ-cell*. The rank of a cell will be indicated by a superscript, $A = A^\rho$.

A cell will be called a *type-cell* if all its terms are negative faces, $z_i = a_i$. The *type* of a cell $A = z_1, z_2, \cdots, z_m$ will be the type-cell

$$(2.2) \qquad \alpha(A) = a_1, a_2, \cdots, a_m$$

of the same degree as A, such that the ith term of the sequence $\alpha(A)$ is the negative face a_i belonging to the same cut as the ith term z_i of A, $i = 1, 2, \cdots, m$. Two cells A and B will be said to be of the same *type* if their type-cells are identical. The number of cells of any given type $\alpha = \alpha(A)$ will, of course, be finite (3^m, to be exact). The type of a cell will be indicated by a subscript, $A = A_\alpha$. Thus, a symbol of the form A_α^ρ will denote a ρ-cell of type α. A cell A will be said to be *regular*, or of *regular type*, if its type $\alpha = \alpha(A)$ is composed of m different terms a_i; it will be said to be *singular*, or of *singular type*, if its type α has any repeated terms. A cell A without repeated terms may, of course, be singular. For example, the cell $A = a_i, c_i$ is singular, since its type $\alpha(A) = a_i, a_i$ is singular.

For the sake of greater formal simplicity, we shall assume that there is an *empty sequence* of elements with no terms at all. Accordingly, we shall have a unique cell E_ϵ of degree $m = 0$. The cell E_ϵ will be called the *unit cell*. It will be of rank $\rho = 0$ and of type $\epsilon = E_\epsilon$. The type ϵ will be called the *unit type*. The unit type will be regular.

Every element z is paired with a cell z of degree $m = 1$ consisting of a single term in the element z. Our notation will make no distinction between the element z and the cell z. We shall call the cell z the *elementary cell* determined by the element z. The type a of an elementary cell z, $z = a$, b, or c, will also be said to be *elementary*.

The *product* AB of a cell $A = z_1, z_2, \cdots, z_m$ and a cell $B = w_1, w_2, \cdots, w_n$ will be the cell

$$(2.3) \qquad AB = z_1, z_2, \cdots, z_m, w_1, w_2, \cdots, w_n$$

obtained by merely appending the sequence B to the sequence A. Thus, the product AB will be of degree $m+n$, of rank $\rho(A)+\rho(B)$, and of type $\alpha\beta$, where α is the type of A and β the type of B. Cell products will be *associative*, but *noncommutative*. The unit cell E_ϵ will be the unit of cell multiplication,

$$E_\epsilon A = A = AE_\epsilon \qquad (A \text{ arbitrary}). \tag{2.4}$$

Generally speaking, it will be preferable to think of a cell A as a product of elementary cells, rather than as a sequence of elements. We shall therefore replace the sequential notation (2.1) by the product notation

$$A = z_1 z_2 \cdots z_m. \tag{2.5}$$

The cells A_α of any fixed type $\alpha = a_1 a_2 \cdots a_m$ can all be represented schematically in the space Ξ^n of n real variables $\xi_1, \xi_2, \cdots, \xi_n, n \geqq m$, by making the following construction. Corresponding to each elementary factor z_i of a cell $A_\alpha = z_1 z_2 \cdots z_m$ we write one of the three relations $\xi_i \leqq 0$, $\xi_i = 0$, $\xi_i \geqq 0$, the first if we have $z_i = a_i$, the second if we have $z_i = b_i$, the third if we have $z_i = c_i$. The m relations thus obtained are regarded as the *determining relations* of the representation of A_α. The *representation* itself consists of all points $\xi = \xi_1, \xi_2, \cdots, \xi_n$ of the space Ξ^n such that their first m coordinates satisfy the determining relations. In Fig. 1, we have represented the three cells a, b, and c of type $\alpha = a$ on the line Ξ^1; in Fig. 2, the same three cells in the plane Ξ^2. In Fig. 3, we have represented the nine cells $z_1 z_2$ of type $\alpha = a_1 a_2$ in the plane Ξ^2.

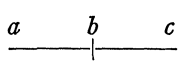

FIG. 1

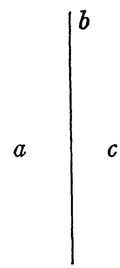

FIG. 2

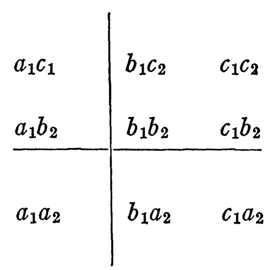

FIG. 3

The representation of a ρ-cell A^ρ in the n-space Ξ^n is evidently a point set of dimensionality $n-\rho$, since the determining relations of the representation consist of ρ independent equations and $n-\rho$ inequalities. The representation of the unit cell E_ϵ in the space Ξ^n is the entire space Ξ^n, since the representation of E_ϵ has no determining relations at all.

Given two *different* cells $A = z_1 z_2 \cdots z_m$ and $B = w_1 w_2 \cdots w_m$ of the same type α, we shall say that the cell A is *on the boundary* of the cell B if, for every i, the factor w_i of B is equal either to the corresponding factor z_i of A or to the edge b_i associated with the factor z_i,

$$(2.6) \qquad A \text{ bd } B \leftrightarrow A \neq B, \quad z_i = w_i \text{ or } b_i, \qquad (i = 1, 2, \cdots, m).$$

Clearly, the cell A will be on the boundary of the cell B if, and only if, the representation of the cell A in the space Ξ^n is made up entirely of points on the geometrical boundary of the representation of the cell B in Ξ^n.

3. **Chains.** Now, let Γ be a grating, let $[A_\alpha]$ be the set of all cells of Γ of some fixed type α, and let $[\lambda]$ be an arbitrary ring of coefficients without divisors of zero. (In most problems, the ring $[\lambda]$ will either be the ring of all the integers or the ring of all the real numbers.) By a *chain* K of the grating Γ we shall mean any mapping of the set $[A_\alpha]$ into the ring $[\lambda]$. The chain K will be said to be of *type* α, where α is the type of the cells A_α. It will be said to be *regular* or *singular* according as its type is regular or singular. We shall write $K = K_\alpha$ when we want to indicate explicitly that the chain K is of type K_α.

In accordance with one of the standard notational procedures, we shall represent a chain K_α by a linear form in the cells of type α,

$$(3.1) \qquad K_\alpha = \sum_i \lambda_i A_{\alpha i}.$$

For every i, the coefficient of the ith term of the form will be the image λ_i of the cell $A_{\alpha i}$ in the mapping K_α. As suggested by the notation, the *sum* and *difference* of a chain $K_\alpha = \sum_i \lambda_i A_{\alpha i}$ and a chain $L_\alpha = \sum_i \mu_i A_{\alpha i}$ will be the chains $K_\alpha + L_\alpha = \sum_i (\lambda_i + \mu_i) A_{\alpha i}$ and $K_\alpha - L_\alpha = \sum_i (\lambda_i - \mu_i) A_{\alpha i}$ respectively. Moreover, a *linear combination* of a finite set of chains $K_{\alpha j} = \sum_i \lambda_{ij} A_{\alpha i}$, $j = 1, 2, \cdots, n$, will be any chain of the form $\sum_j \mu_j K_{\alpha j} = \sum_{ij} \mu_j \lambda_{ij} A_{\alpha i}$, where the coefficients μ_j and λ_{ij} belong to the ring $[\lambda]$. Of course, sums, differences, and linear combinations will not be defined unless all the chains involved are of the same type α.

The cells *of* a chain K_α will be the cells $A_{\alpha i}$ such that their coefficients λ_i in the linear form (3.1) are different from 0. To each type α there will correspond a chain 0_α with no cells at all, such that the coefficients of all the cells $A_{\alpha i}$ are 0. The chain 0_α will be called the *zero* of type α. It will, of course, be a different entity from the zero 0_β of type β, $\beta \neq \alpha$.

To simplify the exposition, we shall assume in the sequel that the ring $[\lambda]$ has a unit element. (The assumption can easily be dispensed with by introducing symbolic chains, in the manner outlined in Remark 3, below.) If there is a unit element $\lambda = 1$, every cell A_α of type α will determine a chain $1 \cdot A_\alpha$ of type α, such that the coefficient of the cell $A_{\alpha i} = A_\alpha$ in (3.1) is 1 and that the coefficients of all the other cells $A_{\alpha i}$ are 0. We shall adopt the standard abbreviation $A_\alpha = 1 \cdot A_\alpha$, and shall say that the chain A_α is the *cell-chain* determined by the cell A_α. Here again, our notation will make no distinction between the cell A_α and the cell-chain A_α. The cell-chain z determined by an elementary cell z will be called an *elementary chain*; the cell-chain E_ϵ determined by the unit cell E_ϵ will be called the *unit chain*. According to (3.1), every chain K_α will be a uniquely determined linear combination of cell-chains $A_{\alpha i}$.

Given two chains $K_\alpha = \sum_i \lambda_i A_{\alpha i}$ and $L_\beta = \sum_j \mu_j B_{\beta j}$ of arbitrary types α and β respectively, we shall say that their *product* is the chain of type $\alpha\beta$ determined by the linear form

$$K_\alpha L_\beta = \sum_{ij} \lambda_i \mu_j \, (A_{\alpha i} B_{\beta j}). \tag{3.2}$$

(The symbol $A_{\alpha i} B_{\beta j}$ in the right-hand member of (3.2) represents the cell product of the cells $A_{\alpha i}$ and $B_{\beta j}$.) Chain multiplication will be associative, and two-way distributive with respect to chain addition. However, it will be noncommutative, since cell multiplication is noncommutative. Clearly, the unit chain E_ϵ will be the unit of chain multiplication,

$$E_\epsilon K_\alpha = K_\alpha = K_\alpha E_\epsilon. \tag{3.3}$$

Moreover, we shall have

$$0_\alpha K_\beta = 0_{\alpha\beta} = L_\alpha 0_\beta. \tag{3.4}$$

A chain K_α will be assigned a definite *degree* m, equal to the degree of its type α. In general, the chain will not be assigned a rank ρ. However, we shall say that a chain K_α is a *ρ-chain*, or that K_α is of *rank* ρ, if no cell of K_α is of any other rank than ρ. According to this definition, a zero 0_α will be of all ranks ρ simultaneously, $\rho = 0, 1, 2, \cdots$, while a cell-chain A_α will be of the same rank as its determining cell A_α. By collecting terms of like rank in (3.1), we shall be able to express a chain K_α in a uniquely determined manner as a sum of the form

$$K_\alpha = \overset{0}{K}_\alpha + \overset{1}{K}_\alpha + \cdots + \overset{m}{K}_\alpha \qquad (m = m(K_\alpha)), \tag{3.5}$$

where $\overset{\rho}{K}_\alpha$, $\rho = 0, 1, 2, \cdots, m$, is a ρ-chain. We shall extend the meaning of the symbol K_α^ρ to higher determinations of the rank ρ by writing $K_\alpha^\rho = 0_\alpha$, for $\rho > m$. The chain K_α^ρ will be called the ρth *constituent* of the chain K_α. We shall treat the superscript ρ as the symbol for an operator—the *rank operator* ρ—transforming a chain K_α into the ρth constituent K_α^ρ of K_α. The operator ρ will, of course, be linear,

$$(3.6) \qquad \left(\sum_i \lambda_i K_{\alpha i}\right)^\rho = \sum_i \lambda_i K_{\alpha i}^\rho,$$

and we shall have

$$(3.7) \qquad \begin{aligned} (K_\alpha^\rho)^\rho &= K_\alpha^\rho, \\ (K_\alpha^\rho)^\sigma &= 0_\alpha \qquad\qquad (\sigma \neq \rho). \end{aligned}$$

Obviously, the product $K^\rho L^\sigma$ of a ρ-chain and a σ-chain will be a $(\rho+\sigma)$-chain.

To avoid confusion, we shall never use the symbol X^ρ, as applied to a cell or chain, to denote the ρ-fold product $XX \cdots X$. The symbol will merely indicate that the cell or chain to which it refers is of rank ρ. One further matter of notation: Instead of expressing a chain K_α as a linear form (3.1) in the cells A_α we can express it as a form of degree m in the elementary cells z by treating each cell A_α as a product of m elementary factors; cf. (2.5). The order of the factors in the terms of the form must, of course, be taken into consideration.

Remark 1. We shall use the standard notation $\gamma \in \Gamma$ to indicate that a cut γ is a member of grating Γ. Moreover, by extending the meaning of the symbol $\in$, we shall write $z \in \Gamma$, $A \in \Gamma$, $K \in \Gamma$ to indicate that z is an element, A a cell, and K a chain of Γ.

Remark 2. As a heuristic device, we can construct a schematic model of a chain K_α by representing the cells $A_{\alpha i}$ of type α in a space Ξ^n, after the manner of §2, and by "weighting" or "labelling" the representation of each cell $A_{\alpha i}$ with the coefficient of $A_{\alpha i}$ in the expression (3.1).

Remark 3. The discussion can obviously be extended to the case in which the ring $[\lambda]$ has no unit elements by immersing the ring $[\lambda]$ in an appropriate ring $[\mu]$ with a unit element. For instance, to state matters rather loosely, we can let $[\mu]$ be the set of all linear expressions $\mu = \lambda + n$ such that λ is a member of the ring $[\lambda]$ and n an integer. The set $[\mu]$ becomes a ring when we combine the expressions μ by addition and multiplication in the manner suggested by the notation $\mu = \lambda + n$. Each element λ of $[\lambda]$ can be identified with the element $\mu = \lambda + 0$ of $[\mu]$. The element $\mu = 0 + 1$ is the unit element of $[\mu]$. A chain K_α is written in the form $K_\alpha = \sum_i \mu_i A_{\alpha i}$, $\mu_i \in [\mu]$. However, it is treated as purely *symbolic* unless all its coefficients μ_i are identified with members of $[\lambda]$, and so on.

4. **The conjugation and boundary operators.** As a formal device for taking care of signs, we shall now define a type-preserving operator, called the *conjugation operator*, acting on the chains of a grating Γ. The conjugation operator will transform the chain K_α represented in (3.5) into the chain

$$K_\alpha^* = \overset{0}{K_\alpha} - \overset{1}{K_\alpha} + \cdots + (-1)^m \overset{m}{K_\alpha}, \tag{4.1}$$

formed by merely changing the sign of all constituents of odd ranks. The chain K_α^* will be called the *conjugate* of the chain K_α. The conjugation operator will evidently be linear,

$$\left(\sum_i \lambda_i K_{\alpha i}\right)^* = \sum_i \lambda_i K_{\alpha i}^*, \tag{4.2}$$

and we shall have

$$K_\alpha^{**} = K_\alpha, \qquad (K_\alpha L_\beta)^* = K_\alpha^* L_\beta^*, \qquad (\overset{\rho}{K_\alpha})^* = (K_\alpha^*)^\rho. \tag{4.3}$$

Next, we shall define a more significant type-preserving operator, called the *boundary operator*. The boundary operator will transform a chain K_α into a chain K_α', called the *boundary* of K_α. We shall require that the boundary operator be linear,

$$\left(\sum_i \lambda_i K_{\alpha i}\right)' = \sum_i \lambda_i K_{\alpha i}'. \tag{4.4}$$

It will therefore be sufficient for us to define the boundaries of the cell-chains. The boundary of a cell-chain A_α will be defined by induction on the degree $m = m(A_\alpha)$ of A_α. As the hypothesis of the induction—which will begin with the value $m = 2$—we shall assume that we know the boundaries of all chains of degree $m-1$.

Case $m = 0$. The only cell-chain of degree 0 is the unit chain E_ϵ. We shall write

$$E_\epsilon' = 0_\epsilon. \tag{4.5}$$

Case $m = 1$. The cell-chains of degree 1 are the elementary chains, which are of the forms $z = a$, b, or c. We shall write

$$a' = b, \qquad b' = 0_a, \qquad c' = -b. \tag{4.6}$$

The zero 0_a will, of course, be the zero of the same type a as the chains a, b, and c.

Case $m>1$. A cell-chain A of degree m, $m>1$, can be written in a uniquely determined manner as the product Bz of a cell-chain B of degree $m-1$ and an elementary chain z. We shall define the boundary of A by the recursion formula

$$A' = B'z + B^*z'. \tag{4.7}$$

The boundary of a general chain K of degree m will then be fully determined, by linearity.

We can easily verify, by induction, that the boundary of a chain is of the same type α as the chain itself, and that the boundary of a ρ-chain is a $(\rho+1)$-chain. Therefore, by (3.5), we can write

$$(K'_\alpha)^0 = 0_\alpha, \qquad (K'_\alpha)^\rho = (K_\alpha^{\rho-1})' \qquad (\rho > 0). \tag{4.8}$$

It is also clear, by (4.1) and (4.8), that the boundary operator anticommutes with the conjugation operator,

$$(K^*)' = -(K')^*. \tag{4.9}$$

The formula for the boundary of a chain product KL is

$$(KL)' = K'L + K^*L'. \tag{4.10}$$

Proof. Since the boundary and conjugation operators are both linear, and since chain multiplication is distributive with respect to chain addition, the proof reduces, at once, to the case in which the factors K and L are both cell-chains. We shall therefore assume that K and L are cell-chains, and shall proceed by induction on the degree m of the second factor L.

The case $m(L)=0$ is trivial. The cell-chain L can only be the unit chain $L=E_\epsilon$. Therefore, we have $L'=0_\epsilon$, and both members of (4.10) reduce to K'.

The case $m(L)=1$ is covered by the assumed recursion formula (4.7).

We treat the case $m(L)>1$ by expressing the cell-chain L as a product $L=AB$, where A and B are cell-chains of degrees less than $m(L)$. By the hypothesis of the induction, we have at once

$$\begin{aligned}(KL)' = (KA\cdot B)' &= (KA)'B + (KA)^*B' \\ &= K'AB + K^*A'B + K^*A^*B' \\ &= K'AB + K^*(AB)' \\ &= K'L + K^*L',\end{aligned}$$

which is what we set out to prove.

The explicit formula for the boundary of a product $K_1K_2 \cdots K_n$ of n factors K_i is

$$(4.11) \quad (K_1K_2 \cdots K_n)' = \sum_i K_1^*K_2^* \cdots K_{i-1}^*K_i' K_{i+1} \cdots K_n,$$

as may be proved directly by induction on n. As a special case of (4.11) we have the formula for the boundary of a cell-chain $A = z_1z_2 \cdots z_m$,

$$(4.12) \quad A' = (z_1z_2 \cdots z_m)' = \sum_i \overset{*}{z_1}\overset{*}{z_2} \cdots \overset{*}{z}_{i-1}z_i'z_{i+1} \cdots z_m.$$

This last formula has a simple geometrical interpretation. The ith term on the right vanishes if we have $z_i = b_i$ (whence, $z_i' = 0$) and is of one of the two forms $\pm z_1z_2 \cdots z_{i-1}b_iz_{i+1} \cdots z_m$ if we have $z_i = a_i$ or c_i. Therefore, the cells *of* the boundary of a ρ-cell A are precisely the $(\rho+1)$-cells B *on* the boundary of A, in the sense of §2.

The boundary of the boundary of a chain K is the zero of the same type α as K,

$$(4.13) \qquad K'' = 0 \qquad \text{(that is, } K_\alpha'' = 0_\alpha\text{)}.$$

Proof. Again, the proof reduces, by linearity, to the case in which the chain K is a cell-chain. We therefore assume that K is a cell-chain, and proceed by induction on the degree m of K.

The case $m=0$ and $m=1$ are both trivial, since we have $E_\epsilon'' = 0_\epsilon$, by (4.5), and $a'' = b'' = c'' = 0_a$, by (4.6).

We treat the case $m > 0$ by expressing the cell-chain K as a product $K = AB$, where A and B are cell-chains of degrees less than m. According to (4.10), we have

$$K' = (AB)' = A'B + A^*B', \qquad K'' = A''B + A^{*\prime}B' + A^{*\prime}B' + A^{**}B'.$$

The two outer terms on the right are zeros, by the hypothesis of the induction; the two inner terms cancel, by (4.9).

5. **Refinors, refinements, pseudo-sums.** The sum of all the cell-chains of type α and rank 0 will be called the *refinor* E_α of type α. The refinor of unit type ϵ will thus be the unit chain E_ϵ, since the only cell of unit type is the cell E_ϵ. The refinor of type $\alpha = a_1a_2 \cdots a_m$, $m > 0$, will be the product

$$(5.1) \qquad E_\alpha = (a_1 + c_1)(a_2 + c_2) \cdots (a_m + c_m).$$

Indeed, if we expand the product on the right we clearly obtain the sum of all cell-chains of the form $A_\alpha = z_1z_2 \cdots z_m$, such that each factor z_i has one of the two determinations a_i or c_i. According to (5.1), the product of two refinors is a refinor,

$$E_\alpha E_\beta = E_{\alpha\beta}. \tag{5.2}$$

Moreover, we have

$$\begin{aligned} &\overset{0}{E_\alpha} = E_\alpha; \qquad \overset{\rho}{E_\alpha} = 0_\alpha \qquad (\rho > 0); \\ &E_\alpha^* = E_\alpha; \qquad E_\alpha' = 0_\alpha. \end{aligned} \tag{5.3}$$

Formulae (5.3) are obvious, with the possible exception of the last one. To prove that the boundary of E_α vanishes, we have only to note that the boundary of each factor (a_i+c_i) vanishes, $(a_i+c_i)' = b_i - b_i = 0_{a_i}$, and to apply (4.11) for the boundary of a product.

A *refinement* of a chain K will be any product of the form KE, where E is a refinor. In view of the identity $K = KE_\epsilon$, a chain K is to be regarded as a refinement of itself. The geometrical meaning of a refinement is as follows. Suppose we construct schematic models of the chains K_α and $K_\alpha E_\beta$ in the same n-space Ξ^n, $n \geqq m(\alpha) + m(\beta)$, after the manner of §3. Then each ρ-cell $A_\alpha^\rho B_\beta^0$ of the refinement $K_\alpha E_\beta$ will be represented by a portion of the geometrical domain representing the corresponding ρ-cell A_α^ρ of K_α. For heuristic purposes, we can think of the chain $K_\alpha E_\beta$ as the one obtained by "subdividing each ρ-cell A_α^ρ of K_α into the subcells $A_\alpha^\rho B_\beta^0$." Thus, in particular, an *elementary refinement* $K(a+c)$ of a chain K will be obtained by "cutting each cell A of K along the edge b, thereby separating the cell A into a pair of subcells Aa and Ac which touch along the edge b."

We can think of a refinor E as an operator transforming a chain K into the refinement KE of K. The operator is linear,

$$\left[\sum_i \lambda_i K_i \right] E = \sum_i \lambda_i (K_i E), \tag{5.4}$$

since chain multiplication is distributive with respect to chain addition. Moreover, by (5.3), the operator E commutes with the rank, conjugation, and boundary operators,

$$(KE)^\rho = K^\rho E, \qquad (KE)^* = K^* E, \qquad (KE)' = K'E. \tag{5.5}$$

Of course, we can also write

$$(EK)^\rho = EK^\rho, \qquad (EK)^* = EK^*, \qquad (EK)' = EK'. \tag{5.6}$$

The sum of two chains is not defined unless the chains are of the same type. To compensate for this defect it will be convenient (though not strictly essential) to introduce a new law of combination

$$K_\alpha \oplus L_\beta = K_\alpha E_\beta + E_\alpha L_\beta, \tag{5.7}$$

which will be applicable to any pair of chains K_α and L_β. The chain $K_\alpha \oplus L_\beta$ will be called the *pseudo-sum* of K_α and L_β. It will be of the same type $\alpha\beta$ as the product $K_\alpha L_\beta$. Similarly, we shall write

$$K_\alpha \ominus L_\beta = K_\alpha \oplus (-L_\beta) = K_\alpha E_\beta - E_\alpha L_\beta, \tag{5.8}$$

and shall call the chain $K_\alpha \ominus L_\beta$ the *pseudo-difference* of K_α and L_β. A chain of the form

$$\begin{aligned}\lambda_1 K_1 \oplus \lambda_2 K_2 \oplus \cdots \oplus \lambda_n K_n \\ = \sum_i \lambda_i E_1 E_2 \cdots E_{i-1} K_i E_{i+1} \cdots E_n\end{aligned} \tag{5.9}$$

will be called a *pseudo-combination* of the chains K_i.

Pseudo-addition will be *associative*,

$$\begin{aligned}K_\alpha \oplus (L_\beta \oplus M_\gamma) = (K_\alpha \oplus L_\beta) \oplus M_\gamma \\ (= K_\alpha E_\beta E_\gamma + E_\alpha L_\beta E_\gamma + E_\alpha E_\beta M_\gamma),\end{aligned} \tag{5.10}$$

but *noncommutative*. In general, the pseudo-difference $K_\alpha \ominus K_\alpha$ will not be zero. According to (5.4), (5.5), and (5.6), the rank, conjugation, and boundary operators will be "pseudo-linear" with respect to pseudo-addition:

$$\begin{aligned}(K \oplus L)^\rho = K^\rho \oplus L^\rho, \qquad (K \oplus L)^* = K^* \oplus L^*, \\ (K \oplus L)' = K' \oplus L'.\end{aligned} \tag{5.11}$$

The following identities are worth keeping in mind:

$$K_\alpha E_\beta = K_\alpha \oplus 0_\beta, \qquad E_\alpha K_\beta = 0_\alpha \oplus K_\beta; \tag{5.12}$$

also the identity

$$\begin{aligned}(K_\alpha + L_\alpha) \oplus (M_\beta + N_\beta) = (K_\alpha \oplus M_\beta) + (L_\alpha \oplus N_\beta) \\ [= (K_\alpha + L_\alpha) E_\beta + E_\alpha (M_\beta + N_\beta)].\end{aligned} \tag{5.13}$$

This last is analogous to a distributive law.

6. **Permutators and permutations.** Let

$$\pi = (i_1, i_2, \cdots, i_n) \tag{6.1}$$

be the permutation on the integers $1, 2, \cdots, n$ which transforms the integer s into the integer i_s ($s = 1, 2, \cdots, n$). Corresponding to the permutation π we shall define a linear operator, or *permutor*—also to be denoted by π—which will transform every chain K of degree m, $m \geqq n$, into a chain πK, called the *permutation* πK of K. Since the operator π is to be linear,

$$\pi\left(\sum_i \lambda_i K_{\alpha i}\right) = \sum_i \lambda_i(\pi K_{\alpha i}) \qquad (m(\alpha) \geqq n), \tag{6.2}$$

it will be sufficient for us to define the permutations $\pi(A)$ of the cell-chains A of degrees m greater than or equal to n. If A is the cell-chain $A = z_1 z_2 \cdots z_m$, $m \geqq n$, we shall write

$$\pi A = (-1)^\sigma (z_{i_1} z_{i_2} \cdots z_{i_n})(z_{n+1} z_{n+2} \cdots z_m), \tag{6.3}$$

where the integer σ which determines the sign of the expression is to be calculated in the following manner. We examine the product $z_1 z_2 \cdots z_m$ and construct all pairs (z_i, z_j) consisting of two different factors $z_i = b_i$ and $z_j = b_j$ of rank 1. (Factors $z_i = a_i$ or c_i of rank 0 are simply ignored.) The integer σ is equal to the number of pairs (z_i, z_j) such that the relative order of z_i and z_j in the product $z_1 z_2 \cdots z_m$ is different from the relative order of z_i and z_j in the product $z_{i_1} z_{i_2} \cdots z_{i_n} z_{n+1} \cdots z_m$. A few examples will make our meaning clear:

(i) Let $A = a_1 b_2 b_3$ be a cell-chain, and let $\pi = (2, 1, 3)$ and $\kappa = (3, 2, 1)$ be permutations on the integers 1, 2, 3. We write $A = b_2 a_1 b_3$, with the *plus* sign, since we have not disturbed the relative order of the factors b_2 and b_3, but we write $\kappa A = -b_3 b_2 a_1$, with the *minus* sign, since we have reversed the order of the factors b_2 and b_3.

(ii) Let $A = bb$ be a singular cell of singular type aa, and let $\pi = (2, 1)$ be the permutation which inverts the integers 1 and 2. We write $\pi A = -bb$, with the *minus* sign, since we have reversed the order of the two factors b. (N.B. A term of a sequence, or a factor of a non-commutative product, is not fully determined unless we know both its *value* and its *position*. When we speak, rather loosely, about identical terms or factors, we are referring, of course, to terms or factors with identical values.)

(iii) Let C be the product of a ρ-cell $A^\rho = z_1 z_2 \cdots z_m$ and a σ-cell $B^\sigma = z_{m+1} z_{m+2} \cdots z_n$, and let κ be the cyclic permutation $(m+1, m+2, \cdots, n, 1, 2, \cdots, m)$. Then we must obviously write

$$\kappa(A^\rho B^\sigma) = (-1)^{\sigma\rho} B^\sigma A^\rho, \tag{6.4}$$

since each of the ρ factors $z_i = b_i$ of A^ρ has undergone an inversion relative to each of the σ factors $z_j = b_j$ of B^σ. By linearity, a similar formula applies to the cyclic permutation κ of the product $K_\alpha^\rho L_\beta^\sigma$ of a ρ-chain K_α^ρ and a σ-chain L_β^σ,

$$\kappa(K_\alpha^\rho L_\beta^\sigma) = (-1)^{\rho\sigma} L_\beta^\sigma K_\alpha^\rho \qquad (\kappa \cdot \alpha\beta = \beta\alpha). \tag{6.5}$$

No such simple formula exists for the permutation $\pi(KL)$ of two general chains K and L. However, if either of the chains K or L is a

0-chain we can, of course, write

$$\kappa(KL) = LK \qquad (K = K^0 \text{ or } L = L^0), \tag{6.6}$$

without any change of sign. In consequence of (6.6), it is clear that the pesudo-sum $L_\beta \oplus K_\alpha = L_\beta E_\alpha + E_\beta K_\alpha$ is a cyclic permutation of the pseudo-sum $K_\alpha \oplus L_\beta = K_\alpha E_\beta + E_\alpha L_\beta$,

$$L_\beta \oplus K_\alpha = \kappa(K_\alpha \oplus L_\beta) \qquad (\kappa \cdot \alpha\beta = \beta\alpha). \tag{6.7}$$

A permutor (6.1) obviously transforms a chain K_α of type $\alpha = a_1a_2 \cdots a_m$, $m > n$, into a chain $K_{\pi\alpha}$ of type

$$\beta = \pi(\alpha) = (a_{i_1}a_{i_2} \cdots a_{i_n})(a_{n+1}a_{n+2} \cdots a_m). \tag{6.8}$$

We shall call the type β the *permutation* $\pi(\alpha)$ of the type α.

Remark. It would be natural to define the permutation of a general cell $A = z_1z_2 \cdots z_m$ by the formula $\pi(A) = (z_{i_1}z_{i_2} \cdots z_{i_n})(z_{n+1}z_{n+2} \cdots z_m)$. However, we prefer to define *only* the permutations of the type-cells α. It would be confusing to have one formula for a permutation of the cell A and a different formula, involving at times a different sign, for a permutation of the corresponding cell-chain A. Formula (6.8) leads to no confusion, whether we interpret the product $a_1a_2 \cdots a_m$ as the symbol for a type-cell or as the symbol for a cell-chain.

We can, of course, write the identity

$$(\pi K_\alpha)L_\beta = \pi(K_\alpha L_\beta), \tag{6.9}$$

provided the degree of K_α is such that the chain πK_α is defined. Similarly, we can write

$$K_\alpha(\pi L_\beta) = \omega(K_\alpha L_\beta) \qquad (\omega \cdot \alpha\beta = \alpha \cdot \pi\beta), \tag{6.10}$$

provided the permutation πL_β is defined. The permutor π obviously commutes with the rank and conjugation operators, so that we are justified in using the simplified notations

$$\pi K^\rho = \pi(K^\rho) = (\pi K)^\rho, \quad \pi K^* = \pi(K^*) = (\pi K)^*. \tag{6.11}$$

The permutor π also commutes with the boundary operator,

$$\pi K' = \pi(K') = (\pi K)', \tag{6.12}$$

though, in this case, a formal proof is necessary because of minor difficulties in regard to signs. Since we are dealing with linear operators, the proof of (6.12) reduces to the case in which K is a cell-chain, $K = z_1z_2 \cdots z_m$. Moreover, every permutation on the integers $1, 2, \cdots, m$ is the resultant of inversions which interchange consecutive integers i and $i+1$. Therefore, the proof reduces, in the last analysis, to the case in which K is a cell-chain of the form $K=AzwB$, and in which π is the permutor which interchanges the adjacent elementary factors z and w. Now, in the case left for consideration, we have

$$\pi K = (-1)^{\rho\sigma} AwzB,$$

where ρ and σ are the ranks of the elementary factors z and w respectively ($\rho=0$ or 1, $\sigma=0$ or 1). Therefore we can write

$$(\pi K)' = (-1)^{\rho\sigma}[A'wzB + A^*w'zB + A^*w^*z'B + A^*w^*z^*B'].$$

On the other hand, we can also write

$$K' = A'zwB + A^*z'wB + A^*z^*w'B + A^*z^*w^*B',$$

whence

$$\pi(K') = (-1)^{\sigma\rho}[A'wzB + (-1)^{\sigma}A^*wz'B + (-1)^{\rho}A^*w'z^*B + A^*w^*z^*B'),$$

since z' and w' are of ranks $\rho+1$ and $\sigma+1$ respectively. To conclude the proof, we have only to note that the expression for $\pi(K')$ is equivalent to the expression for $(\pi K)'$, since we have $z^*=(-1)^{\rho}z$ and $w^*=(-1)^{\sigma}w$.

7. **Grating representations; loci.** A *representation* (Γ, f, X) of a grating $\Gamma=[\gamma]$ on a point set $X=[x]$ will consist of the grating Γ, the set X, and an arbitrary function $f(\gamma, x)$ of a variable cut of Γ and a variable point of X, such that the only allowable values of the function are -1, 0, and 1. The set X will be called the *carrier*, and the function $f(\gamma, x)$ the *determining function* of the representation. If we have a representation (Γ, f, X) of Γ on X, every cut $\gamma_i=(a_i, b_i, c_i)$ of the grating Γ will determine a separation of the carrier X into a trio of disjoint sets A_i, B_i, and C_i, consisting of the points x at which the determining function $f(\gamma_i, x)$ takes on the values -1, 0, and 1 respectively. We shall call the sets A_i, B_i, and C_i the *representatives* of the elements a_i, b_i, and c_i respectively.

Example A. Let X be a topological space, and let $[\gamma]$ be the set of all real, continuous functions $\gamma(x)$ of a variable point x of X. We treat each member γ_i of $[\gamma]$ as the symbol for a cut consisting of three abstract elements a_i, b_i, and c_i. The cuts γ_i form a grating $\Gamma=[\gamma]$. Moreover, we obtain a representation (Γ, f, X) of the grating Γ on the carrier X by letting the determining function $f(\gamma, x)$ be the function which takes on the value -1, 0, or 1 according as the function $\gamma(x)$ is negative, zero, or positive at x. The representatives of the elements a_i, b_i, and c_i of γ_i are, of course, the point sets determined by the relations $\gamma_i<0$, $\gamma_i=0$, and $\gamma_i>0$ respectively. Other significant representations will be given in §13.

Given a representation (Γ, f, X) of a grating Γ on a set X, there is a natural way of assigning to each chain K of Γ an appropriate subset $|K|$ of X. The subset $|K|$ will be called the *locus* of the chain K. It will be defined in the following manner:

To the unit chain E_ϵ, we shall assign the entire set X,

$$(7.1) \qquad |E_\epsilon| = X.$$

To the elementary chains $z_i=a_i$, b_i, and c_i, we shall assign the sets

$$(7.2) \qquad |a_i| = A_i \cup B_i, \qquad |b_i| = B_i, \qquad |c_i| = C_i \cup B_i$$

respectively, where A_i, B_i, and C_i are the representatives of the elements a_i, b_i, and c_i respectively. Thus, in Example A above, the loci $|a_i|$, $|b_i|$, and $|c_i|$ will be the sets determined by the conditions $\gamma_i\leqq 0$, $\gamma_i=0$, and $\gamma_i\geqq 0$ respectively. According to (7.2), we shall have

$$(7.3) \qquad |a_i| \cup |c_i| = X, \qquad |a_i| \cap |c_i| = |b_i|.$$

The locus of a cell-chain $A=z_1z_2\cdots z_m$ of degree m greater than 1 will be defined to be the intersection of the loci of the elementary factors of A,

$$(7.4) \qquad |A| = |z_1| \cap |z_2| \cap \cdots \cap |z_m|.$$

The formula for the product of two cell-chains A and B will thus be

$$(7.5) \qquad |AB| = |A| \cap |B|.$$

Moreover, we shall have

$$(7.6) \qquad A \text{ bd } B \rightarrow |A| \subseteq |B|.$$

Indeed, if the cell $A=z_1z_2\cdots z_m$ is on the boundary of the cell $B=w_1w_2\cdots w_m$, we can write $z_i=w_i$ or b_i, by (2.6); whence $|z_i| \subseteq |w_i|$, by (7.3); whence, finally, $|A|=\bigcap_i|z_i|\subseteq\bigcap_i|w_i|=|B|$, by (7.4). The locus of a multiple λA of a cell-chain A will be said to be identical with the locus of A, provided the coefficient λ does not vanish. If the coefficient vanishes, the locus will be said to be the empty set,

$$(7.7) \qquad |\lambda A| = |A| \quad (\lambda \neq 0); \qquad |0\cdot A| = 0.$$

The following identities will be useful in the sequel:

$$(7.8) \qquad |ac| = |ab| = |bc| = |bb| = |b|.$$

They are immediate consequences of (7.2) and (7.4).

To complete the definition of the locus, we shall say that the locus of a general chain $K=\sum_i \lambda_i A_i$ is the union of the loci of the terms of K,

$$(7.9)\qquad |K| = |\lambda_1 A_1| \cup |\lambda_2 A_2| \cup \cdots \cup |\lambda_n A_n|.$$

We can evidently write the identities

$$(7.10)\qquad |K| = |K^*| = |\pi K|,$$

since corresponding terms of K, K^*, and πK can differ, at most, in sign and in the arrangement of the elementary factors z_i within the terms. We can also write

$$(7.11)\qquad |\lambda K + \mu L| \subseteq |K| \cup |L|,$$

since every nonvanishing term of $\lambda K+\mu L$ corresponds to a similar nonvanishing term of one, at least, of the chains K and L. Again, we can write

$$(7.12)\qquad |KL| = |K| \cap |L|,$$

since the nonvanishing terms of KL are the products of the nonvanishing terms of K and the nonvanishing terms of L. The locus of a zero is, of course, empty.

Remark. In proving the equality (7.12), we make use, for the first time, of the assumption that the ring of coefficients $[\lambda]$ has no zero divisors. If the ring were allowed to have zero divisors, the product of a nonvanishing term of K and a nonvanishing term of L might be a vanishing term of KL, so that the equality (7.12) would have to be replaced by the inequality $|KL| \subseteq |K| \cap |L|$.

According to (7.3) and (7.9), the locus of an elementary refinor a_i+c_i is $|a_i+c_i| = X$. Therefore, by (7.12), we can write

$$(7.13)\qquad |E| = X, \qquad |KE| = |EK| = |K|,$$

where E is an arbitrary refinor and K an arbitrary chain. As a corollary to (7.11) and (7.13), we have

$$(7.14)\qquad |\lambda K \oplus \mu L| \subseteq |K| \cup |L|.$$

Finally, by (7.6) and (7.9),

$$(7.15)\qquad |K'| \subseteq |K|,$$

since every cell of K' is on the boundary of some cell of K.

In the applications to topology, the carrier of a grating Γ will ordinarily be a topological space. A representation of a grating Γ on a topological space X will be said to be *continuous* provided the representatives A_i and C_i of all the faces of Γ are open sets. If the representation is continuous, the representatives B_i of the edges must be closed sets, since the sets B_i are the complements of the open sets $A_i \cup C_i$. The loci of the chains K of Γ are also closed sets, as may be verified immediately by considering successively the loci (7.1), (7.2), (7.4), and (7.9). The representation (Γ, f, X) in Example A above is evidently continuous.

Remark. A representation (Γ, f, X) of a grating Γ on a point set X induces a topology on the set X. When we speak of the *natural topology* induced by (Γ, f, X) on X we shall mean the coarsest topology such that the representatives of all the faces of Γ are open sets—that is, the coarsest topology such that the representation of Γ on X is continuous. With reference to the natural topology induced on X, we can construct a basic set of neighborhoods consisting of all sets of the form $Z_{i_1} \cap Z_{i_2} \cap \cdots \cap Z_{i_n}$ (n finite, but not fixed), such that the Z_i's are the representatives of faces of Γ, $Z_i = A_i$ or C_i. In other words, we can construct a sub-basic set of neighborhoods consisting of the representatives Z_i of all the faces of Γ.

8. **The intrinsic representation of a grating.** An *intrinsic point* y of a grating Γ will be any single-valued function $y = y(\gamma)$ of the cuts of Γ, such that the only allowable values of the function are -1, 0, and 1. The set $Y = [y]$ of all intrinsic points of Γ will be called the *intrinsic carrier* of the grating. The *intrinsic representation* of the grating will be the representation (Γ, h, Y) of Γ on Y determined by a function $h(\gamma, y)$, such that the value of the function, for every fixed determination of γ and y, is equal to the value of the function $y(\gamma)$ for the given determination of γ, $h(\gamma, y) = y(\gamma)$. The representative of an element z of Γ in relation to the intrinsic representation (Γ, h, Y) will be called the *intrinsic representative* of z, the locus of a chain K will be called the *intrinsic locus* of K. Clearly, the intrinsic representatives of the elements $z_i = a_i$, b_i, and c_i will be the sets made up of all functions (intrinsic points) $y(\gamma)$ such that we have $y(\gamma_i) = -1$, 0, and 1 respectively. The intrinsic locus of a chain K will be denoted by a special symbol $\|K\|$.

The intrinsic representation of a grating is of some theoretical interest, in that it enables us to describe the properties of the grating in pseudo-combinatory language, without reference to limits and continuity. However, it can ordinarily be dispensed with in problems involving a prescribed representation of the grating on a point set X. When we want to talk about the intrinsic representation in topological terms, we shall assign the natural topology to the intrinsic carrier Y (cf. §7). Thus, the intrinsic representation will be continuous.

The chains K of a grating Γ will be partially ordered in terms of their intrinsic loci, as follows. If the intrinsic locus $\|K\|$ of a chain K is contained in the intrinsic locus $\|L\|$ of a chain L, $\|K\| \subseteq \|L\|$, we shall say that K is *dominated* by L and that L *dominates* K. The relations of dominating and of being dominated are evidently transitive, since the relation of inclusion is transitive. However, each of two different chains may dominate the other (cf., for example, the chains ac, ab, and so on, in (7.8)), so that the relations are not anti-symmetrical.

Remark The question of whether or not a prescribed chain K is dominated by a prescribed chain L can always be answered effectively, by a finite process. Consider, first, the case in which the chains are cell-chains, $K=A$, $L=B$. The locus $\|A\|$ consists of all intrinsic points $y(\gamma)$, such that the functions $y(\gamma)$ satisfy a finite set of relations, each of one of the forms $y(\gamma_i) \leqq 0$, $y(\gamma_i)=0$, or $y(\gamma_i) \geqq 0$. A similar remark applies to the locus $\|B\|$. Therefore, by comparing the two sets of relations, for $\|A\|$ and for $\|B\|$, we can determine, by inspection, whether or not there exists an intrinsic point belonging to the locus $\|A\|$, but not to the locus $\|B\|$. Consider, next, the general case. The chain K has only a finite number of cells A, the chain L a finite number of cells B. Therefore, since we know how to handle the special case, we can always determine effectively whether or not there exists an intrinsic point belonging to some $\|A\|$, but not to any $\|B\|$. In other words, we can determine effectively whether or not we have $\|K\| \subseteq \|L\|$.

[8.1] THEOREM. *A representation* (Γ, f, X) *of a grating* $\Gamma=[\gamma]$ *on a space* X *induces a mapping* $y=\mu(x)$ $[=f(\gamma, x)]$ *of the space* X *into the intrinsic carrier* Y *of* Γ. *Moreover, the mapping of* X *into* Y *is continuous if the representation* (Γ, f, X) *is continuous.*

PROOF. The first part of the theorem is obvious if the notation is clearly understood. For a fixed determination $\bar{x}$ of x, the determining function $f(\gamma, x)$ of the representation (Γ, f, X) becomes a function $\bar{y}=f(\gamma, \bar{x})$ of γ alone, with values limited to -1, 0, and 1. Thus, the function $\bar{y}$ is an intrinsic point of Γ, by the very definition of the intrinsic points. By the transformation $y=\mu(x)$, we mean the one which carries the point $\bar{x}$ into the point $\bar{y}$. To verify that the transformation is continuous when (Γ, f, X) is continuous, we let z_i be an element of Γ, Z_X the representation of z_i in X, and Z_Y the representation of z_i in Y. The set Z_Y consists of all intrinsic points $y=y(\gamma)$ such that we have $y(\gamma_i)=k$ ($k=-1$, 0, or 1 according as we have $z_i=a_i$, b_i, or c_i). The set Z_X consists of all points x of X such that we have $f(\gamma_i, x)=k$.

In other words, the set Z_X consists of all points of X which are carried into points of Z_Y by the transformation $y=\mu(x)$ $[=f(\gamma, x)]$. The continuity of the transformation is now obvious. Indeed, the intrinsic representations Z_Y of the *faces* z_i of Γ constitute a sub-basic set of neighborhoods of Y (cf. §7); the inverse images $Z_X=\mu^{-1}(Z_Y)$ of the Z_Y's are open, since the representation (Γ, f, X) is continuous; therefore, by a well known criterion for continuity, the transformation $y=\mu(x)$ is continuous.

[8.2] COROLLARY. *Let (Γ, f, X) be a representation of a grating Γ on a point set X, and let K and L be chains of Γ. Then if the chain K is dominated by the chain L the locus of K is contained in the locus of L,*

$$(8.1) \qquad \|K\| \subseteq \|L\| \rightarrow |K| \subseteq |L|.$$

PROOF. In the notation of the theorem, we have $|E_\epsilon| = \mu^{-1}\|E_\epsilon\|$, by (7.1), $|z| = \mu^{-1}\|z\|$, by (7.2), $|A| = \mu^{-1}\|A\|$, by (7.4), and finally $|K| = \mu^{-1}\|K\|$, by (7.9). Thus, every point x of $|K|$ is mapped into a point $y=\mu(x)$ of $\|K\|$. Now suppose we have $\|K\|\subseteq\|L\|$. Then the image y of x must also be a point of $\|L\|$. Therefore, the point x must be a point of $|L|$. In other words, we must have $|K|\subseteq|L|$.

9. **Grating ideals. Cycles and bounding cycles.** An *ideal* Φ of a grating Γ will be any set of chains of Γ with the following three properties:

(i) The zero 0_ϵ of unit type ϵ is a member of the set.

(ii) The sum $K_\alpha+L_\alpha$ of two members of the set of like type α is a member of the set.

(iii) Every chain K dominated by a member of the set is a member of the set.

[9.1] THEOREM. *Let Φ be an ideal of a grating Γ. Then:*

(α) *All the zeros 0_α of Γ are members of Φ.*

(β) *If K is a member of Φ, so also are λK, K^ρ, K^*, πK, KE, and K'. Moreover, the cell-chains A determined by the cells of K are members of Φ.*

(γ) *If K is a member of Φ and L an arbitrary chain of Γ, the chains KL and LK are members of Φ.*

(δ) *Every linear combination of members of Φ is a member of Φ; every pseudo-combination of members of Φ is a member of Φ.*

The proof is trivial. Statement (α) follows from (i) and (iii), since $\|0_\alpha\| = \|0_\epsilon\|$ is the empty set. Statement (β) follows from (iii), by (7.9), (7.10), (7.13), and (7.15). Statement (γ) follows from (iii), by (7.12). Finally, Statement (δ) follows from (ii) and (β).

Given a grating Γ, there is a smallest ideal of Γ consisting of the chains 0_α only, also a largest ideal consisting of all the chains of Γ. We shall denote the smallest ideal by the symbol Ω, and the largest ideal by the same symbol Γ as the grating itself. Obviously, the intersection $\bigcap_i \Phi_i$ of the members Φ_i of an arbitrary family of ideals of Γ is an ideal of Γ. Thus, if $[K]$ is any set of chains of Γ, there is a smallest ideal Φ of Γ such that we have $[K]\subseteq\Phi$. The ideal Φ is, of course, the intersection of all the ideals Φ_i of Γ such that we have $[K]\subseteq\Phi_i$. The *join* of a family of ideals Φ will be the smallest ideal Φ such that we have $\Phi_i\subseteq\Phi$, for all members Φ of the family.

A chain K of Γ will be said to be a *cycle, modulo* an ideal Φ, if the boundary K' of K is a member of Φ,

$$K' = Z \qquad (Z \in \Phi). \tag{9.1}$$

If a chain K belonging to an ideal Ψ is a cycle, modulo an ideal Φ, we shall say that K is a *cycle* of Ψ, *modulo* Φ. A chain K will be called an *absolute cycle* if its boundary is a zero,

$$K' = 0. \tag{9.2}$$

The theory of absolute cycles does not require special treatment, since an absolute cycle is merely a cycle, modulo the ideal Ω consisting of the zeros only.

[9.2] THEOREM. *Let Φ be an ideal of a grating Γ. Then:*

(α) *All the zeros 0_α and all the refinors E_α of Γ are cycles, modulo Φ.*

(β) *If K is a cycle, modulo Φ, so also are λK, K^ρ, K^*, πK, KE, and K'.*

(γ) *If K and L are cycles, modulo Φ, so also is their product KL.*

(δ) *Every linear combination of cycles, modulo Φ, is a cycle, modulo Φ.*

(ϵ) *Every pseudo-combination of cycles, modulo Φ, is a cycle, modulo Φ.*

The theorem follows, at once, from the theorems on loci in §7. We leave the routine verification to the reader.

Now, let Φ and Ψ be ideals of a grating Γ. A chain K of Γ will be said to *bound* in Ψ, *modulo* Φ, if it can be written in the form

$$K = W' + Z \qquad (W \in \Psi, Z \in \Phi). \tag{9.3}$$

By applying the boundary operator to the two members of (9.3), we obtain an expression of the form (9.1), since the chain W'' is always a zero. Therefore, a chain of Ψ which bounds in Ψ, modulo Φ, is always a cycle of Ψ, modulo Φ.

[9.3] THEOREM. *Let Φ and Ψ be ideals of a grating Γ. Then:*

(α) *All the zeros 0_α of Γ bound in Ψ, modulo Φ.*

(β) *If K is a cycle which bounds in Ψ, modulo Φ, so also are λK, K^ρ, K^*, πK, KE, and K'.*

(γ) *If K is a cycle which bounds in Ψ, modulo Φ, and if L is an arbitrary cycle of Γ, modulo Φ, the products KL and LK are cycles bounding in Ψ, modulo Φ.*

(δ) *Every linear combination of cycles bounding in Ψ, modulo Φ, is a cycle bounding in Ψ, modulo Φ.*

(ϵ) *Every pseudo-combination of cycles bounding in Ψ, modulo Φ, is a cycle bounding in Ψ, modulo Φ.*

Again, the verification of the theorem is immediate. We shall merely give the proof of Statement (γ) which involves a minor difficulty.

According to (9.3) and (9.1), the bounding cycle K can be written in the form $K = W' + Z$, and the boundary of the cycle L in the form $L' = \overline{Z}$ ($Z, \overline{Z} \in \Phi$, $W \in \Psi$). We can therefore write

$$KL = W' L + ZL = (WL)' - W^*\overline{Z} + ZL.$$

Now, by [9.1], the chain WL is necessarily a chain of Ψ, since W is a chain of Ψ. Moreover, the chain $-W^*\overline{Z} + ZL$ is a chain of Φ, since $\overline{Z}$ and Z are chains of Φ. Therefore, the chain KL is of the form (9.3). In other words, the chain KL bounds in Ψ, modulo Φ. The proof that the chain LK bounds in Ψ, modulo Φ, is treated in a similar manner.

We shall introduce the notation

$$(9.4) \qquad K \approx 0, \qquad (\Psi/\Phi),$$

to indicate that K is a chain which bounds in Ψ, modulo Φ. If a pseudo-combination bounds in Ψ, modulo Φ,

$$(9.5) \qquad \lambda_1 K_1 \oplus \lambda_2 K_2 \oplus \cdots \oplus \lambda_n K_n \approx 0, \qquad (\Psi/\Phi),$$

we can rearrange the terms of the left-hand member of (9.5) in any order we please, since the effect of rearranging the terms is merely to transform the pseudo-combination $\lambda_1 K_1 \oplus \lambda_2 K_2 \oplus \cdots \oplus \lambda_n K_n$ into a permutation of itself.

10. **A permutation theorem.** According to §6, the permutors π are linear operators which commute with the rank, conjugation, and boundary operators. It would therefore be feasible to identify a chain K with its permutations.[2] We prefer, however, to let matters stand as they are, and to distinguish between the chains K and πK unless,

[2] A similar identification is made in the classical theory of complexes of simplexes, where an oriented simplex is treated as a skew-symmetrical function of its $n+1$ vertices.

of course, they happen to be formally identical. It is of some theoretical interest that, in view of the following theorem and its corollary, we can develop a satisfactory homology theory without having to make the identifications.

[10.1] THEOREM. *Let K be a chain of a grating Γ, and let πK be any permutation of K, such that the chains K and πK are of the same type α. Then the difference between the chains K and πK can always be expressed in the form*

$$K - \pi K = W' + Z \qquad (\|W\| \subseteq \|K\|, \|Z\| \subseteq \|K'\|), \tag{10.1}$$

where W is a chain dominated by K and Z a chain dominated by the boundary of K.

PROOF. The theorem is trivial unless the chain K is singular. Indeed, if K is of regular type α, the only permutation of K of type α is K itself. Therefore, the difference $K-\pi K$ can be put in the form (10.1) by writing $W=Z=0_\alpha$.

If the type α of K is singular, the theorem reduces, without difficulty, to the special case in which the type of K is of the form $\alpha=aa\beta$, and in which the permutor π is the *invertor* (2, 1) which merely interchanges the first two elementary factors of $aa\beta$. Let us therefore first dispose of this special case. By grouping together the terms of K with the same first two elementary factors, we can express K in the form

$$\begin{aligned} K = \quad & aaK_{11} + abK_{12} + acK_{13} \\ & + baK_{21} + bbK_{22} + bcK_{23} \\ & + caK_{31} + cbK_{32} + ccK_{33}, \end{aligned} \tag{10.2}$$

where the K_{ij}'s are chains of type β. Moreover, in the same notation, we can write

$$\begin{aligned} \pi K = \quad & aaK_{11} + abK_{21} + acK_{31} \\ & + baK_{12} - bbK_{22} + bcK_{32} \\ & + caK_{13} + cbK_{23} + ccK_{33}. \end{aligned} \tag{10.3}$$

(Note the sign of the term $-bbK_{22}$.) The chain $K-\pi K$ is therefore of the form

$$K - \pi K = (ab - ba)P + (ac - ca)Q + (bc - cb)R + 2bbS, \tag{10.4}$$

where P, Q, R, and S are appropriate linear combinations of the K_{ij}'s ($P=K_{12}-K_{21}$, and so on). We shall prove that Condition (10.1) is fulfilled when W and Z are chosen in the following manner:

(10.5) $W = (ab - ba)S + (ac - ca)R, \qquad Z = K - \pi K - W'$,

where the symbols S and R have the same meaning as in (10.4). From the very form of the chain Z, it is clear that W and Z satisfy the *equation* (10.1). What we still have to prove is that W is dominated by K and that Z is dominated by the boundary of K.

Now, it is evident, by inspection, that no two of the chains abP, $-baP$, acQ, $-caQ$, and so on, in the right-hand member of (10.4) can have a cell in common, since the cells of abP all have the left-hand factor ab, the cells of $-baP$ the left-hand factor ba, and so on. We can therefore write the *equality*

$$\|K - \pi K\| = \|abP\| \cup \|-baP\| \cup \|acQ\| \cup \cdots \cup \|2bbS\|.$$

By a similar argument, applied to (10.5), we can also write the equality

$$\|W\| = \|abS\| \cup \|-baS\| \cup \|acR\| \cup \|-caR\|.$$

Furthermore, in view of (7.8) and (7.4), the terms abS and $-baS$ of W are dominated by the term $2bbS$ of $K-\pi K$, and the terms acR and $-caR$ of W are dominated by the term bcR of $K-\pi K$,

$$\|abS\| = \|-baS\| = \|2bbS\| = \|b\| \cap \|S\|,$$
$$\|acR\| = \|-caR\| = \|bcR\| = \|b\| \cap \|R\|.$$

We can therefore write

$$\|W\| \subseteq \|K - \pi K\| \subseteq \|K\| \cup \|\pi K\| \subseteq \|K\|,$$

which proves that the chain W is dominated by the chain K. (*Remark.* The proof makes use of the fact that the ring of coefficients $[\lambda]$ has no zero divisors, since we have to assume $\|2bbS\| = \|bbS\|$.)

To complete the proof for the special case under consideration, we have only to show that the chain Z is dominated by the chain K'. According to (10.5), we obtain, by direct calculation,

$$(10.6) \qquad \begin{aligned} W' = &- (ab - ba)(R + S') + (ac - ca)R' \\ &+ (bc - cb)R + 2bbS. \end{aligned}$$

Therefore, by (10.4) and (10.6), the chain Z is of the form

$$(10.7) \qquad \begin{aligned} Z = K - \pi K - W' &= (ab - ba)M + (ac - ca)N \\ &\qquad (M = P + S' + R,\ N = Q - R'). \end{aligned}$$

Next, we construct the boundary Z' of Z. Since the expression (10.7) for Z is similar in form to the expression (10.5) for W, we can write

$$(10.8)\qquad \begin{aligned} Z' = &- (ab - ba)(N + M') + (ac - ca)N' \\ &+ (bc - cb)N + 2bbM, \end{aligned}$$

without formal computation, by comparison with the expression (10.6) for W'. Finally, by an argument similar to the one made a moment ago, we note that the terms abM and $-baM$ of Z are dominated by the term $2bbM$ of Z', and that the terms acN and $-caN$ of Z are dominated by the term bcN of Z'. Therefore, we can write

$$\|Z\| \subseteq \|Z'\| = \|K' - \pi K'\| \subseteq \|K'\| \cup \|\pi K'\| \subseteq \|K'\|.$$

This completes the proof for the special case under consideration.

The rest of the argument is trivial. We next consider the somewhat more general case in which the type α of K is of the form $\alpha = \beta a \gamma a \delta$, with two identical factors a, and in which the permutor π is the invertor which merely interchanges the two factors a. Let κ be a permutor which transforms the type α into the type $\kappa\alpha = aa\beta\gamma\delta$, and let ω be the invertor (2, 1). Then, according to the case already considered, we can write

$$\kappa K - \omega(\kappa K) = W' + Z \qquad (\|W\| \subseteq \|\kappa K\|, \|Z\| \subseteq \|\kappa K'\|).$$

Thus, we have, at once,

$$K - \pi K = \kappa^{-1}[\kappa K - \omega(\kappa K)] = \kappa^{-1}[W' - Z],$$

$$(\|\kappa^{-1}W\| = \|W\| \subseteq \|\kappa K\| = \|K\|, \|\kappa^{-1}Z\| = \|Z\| \subseteq \|\kappa K'\| = \|K'\|),$$

as required.

The proof of the theorem for the completely general case follows directly from the fact that every permutor π which leaves the type α invariant is the resultant of invertors of the kind just considered.

[10.2] Corollary. *Let Φ and Ψ be ideals of a grating Γ and let K be any cycle of Ψ, modulo Φ. Then if πK is any permutation of K the pseudo-difference $K \ominus \pi K$ bounds in Ψ, modulo Φ:*

$$(10.9)\qquad K \ominus \pi K \approx 0, \qquad (\Psi/\Phi).$$

Proof. The left-hand member of (10.9) can be written $K \cdot \pi E - E \cdot \pi K$, where E is the refinor of the same type type as K. Moreover, the chain $E \cdot \pi K$ is a permutation of the chain $K \cdot \pi E$. Therefore, by the theorem, we can write

$$(10.10)\qquad \begin{gathered} K \cdot \pi E - E \cdot \pi K = W' + Z, \\ (\|W\| \subseteq \|K \cdot \pi E\| = \|K\|, \|Z\| \subseteq \|(K \cdot \pi E)'\| = \|K'\|). \end{gathered}$$

Now, the chain W belongs to the ideal Ψ, since it is dominated by the member K of Ψ, and the chain Z belongs to the ideal Φ, since it is dominated by the boundary K' of the cycle K of Ψ, modulo Φ, which boundary K' is, of course, a chain of Φ. Thus, the desired relation (10.9) is implied by (10.10).

11. **Homologies, homology groups, and homology rings.** The notion of homology is closely related to the notion of bounding. Let Φ and Ψ be ideals of a grating Γ, and let K_α and L_β be chains of Γ. We shall say that the chain K_α is *homologous* to the chain L_β in Ψ, *modulo* Φ,

$$(11.1) \qquad K_\alpha \sim L_\beta, \qquad (\Psi/\Phi),$$

if there exists a refinement of the pseudo-difference $K_\alpha \ominus L_\beta$ such that the refinement bounds in Ψ, modulo Φ,

$$(11.2) \qquad (K_\alpha E_\beta - E_\alpha L_\beta)E_\gamma = K_\alpha \ominus L_\beta \oplus 0_\gamma \approx 0, \qquad (\Psi/\Phi).$$

The relation of homology is clearly *symmetrical*,

$$(11.3) \qquad K_\alpha \sim L_\beta \rightarrow L_\beta \sim K_\alpha, \qquad (\Psi/\Phi),$$

since the relation (11.2) goes over into the relation $L_\beta \ominus K_\alpha \oplus 0_\gamma \approx 0$, by a permutation and a change of sign. In order that K_α be homologous to L_β it is *sufficient*, though not necessary, that the pseudo-difference $K_\alpha \ominus L_\beta$ bound, since we can then write

$$(11.4) \qquad K_\alpha \ominus L_\beta = K_\alpha \ominus L_\beta \oplus 0_\epsilon \approx 0, \qquad (\Psi/\Phi).$$

Another useful condition, which is both necessary and sufficient, is that there be a chain Y_γ of Γ satisfying the joint relations

$$(11.5) \qquad K_\alpha \ominus L_\beta \oplus Y_\gamma \approx 0, \qquad Y_\gamma \approx 0, \qquad (\Psi/\Phi).$$

Indeed, the fulfillment of (11.2) implies the fulfillment of (11.5) with Y_γ set equal to 0_γ. On the other hand, the fulfillment of the first part of (11.5) implies that we have

$$(11.6) \qquad (K_\alpha \ominus L_\beta \oplus 0_\gamma) + (0_\alpha \oplus 0_\beta \oplus Y_\gamma) \approx 0, \qquad (\Psi/\Phi),$$

by (5.13). Moreover, the fulfillment of the second part of (11.5) implies the relation $0_\alpha \oplus 0_\beta \oplus Y_\gamma \approx 0$, since $0_\alpha \oplus 0_\beta \oplus Y_\gamma$ is a permutation of a refinement of Y_γ. By combining this relation with (11.6), we obtain (11.2).

In the remainder of this section, we shall confine our attention to cycles $K, L, M, \cdots$ of Ψ, modulo Φ, as a result of which the theory will assume a particularly simple form. In the first place, it is clear that a cycle K of Ψ, modulo Φ, is homologous to all its permutations,

$$(11.7)\qquad K_\alpha \sim \pi K_\alpha, \qquad (\Psi/\Phi),$$

by (10.9) and (11.4). Thus, in particular, the homology relation, as applied to cycles of Ψ, modulo Φ, is *reflexive,*

$$(11.8)\qquad K_\alpha \sim K_\alpha, \qquad (\Psi/\Phi).$$

We can further verify that the homology relation, as applied to cycles, is *transitive,*

$$(11.9)\qquad K_\alpha \sim L_\beta,\ L_\beta \sim M_\gamma \rightarrow K_\alpha \sim M_\gamma, \qquad (\Psi/\Phi).$$

Indeed, the relations $K_\alpha \ominus L_\beta \oplus 0_\mu \approx 0$ and $L_\beta \ominus M_\gamma \oplus 0_\nu \approx 0$ together imply the relation $K_\alpha \ominus M_\gamma \oplus (L_\beta \ominus L_\beta \oplus 0_\mu \oplus 0_\nu) \approx 0$, by pseudo-addition and permutation. Moreover, the refinement $L_\beta \ominus L_\beta \oplus 0_\mu \oplus 0_\nu$ of $L_\beta \ominus L_\beta$ is a bounding cycle, by (10.9). Therefore, we have $K_\alpha \sim M_\gamma$, by (11.5).

In view of (11.8), (11.3), and (11.9), the cycles K of Ψ, modulo Φ, fall into mutually exclusive *homology classes* $[K]$, such that two cycles belong to the same class if, and only if, they are homologous in Ψ, modulo Φ,

$$(11.10)\qquad [K] = [L] \leftrightarrow K \sim L, \qquad (\Psi/\Phi).$$

The permutations πK of a cycle K all belong to the same class $[K]$, by (11.7). So also do the refinements $KE = K \oplus 0$ of K, since we have $K \ominus (K \oplus 0) \approx 0$, by (10.9). The class $[0_\epsilon]$, which we shall call the *zero class,* will consist of all cycles K_α such that some refinement $K_\alpha \oplus 0_\gamma$ of K_α bounds in Ψ, modulo Φ. To verify this, we only have to write $L_\beta = 0_\epsilon$ in (11.2). In particular, the class $[0_\epsilon]$ will include all the zeros 0_α of Γ and all the chains of $\Phi \cap \Psi$.

The *constituents, conjugate,* and *boundary* of a class $[K]$ will be defined by the formulae

$$(11.11)\qquad [K]^\rho = [K^\rho], \quad [K]^* = [K^*], \quad [K]' = [K'] = [0_\epsilon].$$

To justify the definitions, we must be sure that the classes $[K^\rho]$, $[K^*]$, and $[K']$ are independent of the choice of the cycle K within its class $[K]$,

$$(11.12)\qquad K \sim \overline{K} \rightarrow K^\rho \sim \overline{K}^\rho, \qquad K^* \sim \overline{K}^*, \qquad K' \sim \overline{K}', \qquad (\Psi/\Phi).$$

The verification is immediate, since the relation $K \ominus \overline{K} \oplus 0 \approx 0$ implies the relations $(K \ominus \overline{K} \oplus 0)^\rho = K^\rho \ominus \overline{K}^\rho \oplus 0 \approx 0$, and so on. The class $[K']$ is, of course, the zero class, since the boundary K' of a cycle of Ψ, modulo Φ, is a chain of Φ.

The *sum* of two homology classes will be defined by the formula

$$(11.13)\qquad [K] + [L] = [K \oplus L].$$

Here again, we must be sure that the class $[K\oplus L]$ is independent of the choice of the chains K and L within their respective classes,

$$(11.14) \qquad K\sim\overline{K}, \quad L\sim\overline{L}\rightarrow K\oplus L\sim\overline{K}\oplus\overline{L}, \qquad (\Psi/\Phi).$$

The proof consists merely in noting that the relations $K\ominus\overline{K}\oplus 0_\mu\approx 0$ and $L\ominus\overline{L}\oplus 0_\nu\approx 0$ imply the relation $(K\oplus L)\ominus(\overline{K}\oplus\overline{L})\oplus 0_{\mu\nu}\approx 0$, by pseudo-summation and permutation. The sum (11.13) is *associative*, since pseudo-summation is associative; it is commutative, since the chains $K\oplus L$ and $L\oplus K$ differ, at most, by a permutation; the class $[0_\epsilon]$ is the *zero* with respect to class addition,

$$(11.15) \qquad [K]+[0_\epsilon]=[K\oplus 0_\epsilon]=[K];$$

the class $[-K]$ is the *negative* of the class $[K]$,

$$(11.16) \qquad [K]+[-K]=[K\ominus K]=[0_\epsilon],$$

by (10.9). In other words, *the homology classes form an additive group.*

The following two theorems will enable us to operate more freely with homologies. Let K_α and L_α be cycles of Ψ, modulo Φ, both of the same type α.

Then we have

$$(11.17) \qquad L_\alpha\approx 0\rightarrow K_\alpha+L_\alpha\sim K_\alpha, \qquad (\Psi/\Phi)$$

and

$$(11.18) \qquad K_\alpha\oplus L_\alpha\sim K_\alpha+L_\alpha, \qquad (\Psi/\Phi).$$

The proof of (11.17) is as follows. According to (11.4), the relation $(K_\alpha+L_\alpha)\ominus K_\alpha\approx 0$ is a sufficient condition that the homology $K_\alpha+L_\alpha\sim K_\alpha$ be true. With the aid of (5.13), we write the relation in the alternative form $(K_\alpha\ominus K_\alpha)+(L_\alpha\oplus 0_\alpha)\approx 0$. The first term on the left bounds, by (10.9); the second term bounds, since $L_\alpha\oplus 0_\alpha$ is a refinement of the bounding chain L_α. Thus, the relation is verified, and (11.17) is established. The second theorem, (11.17), follows immediately from the identity

$$K_\alpha\oplus L_\alpha=[(K_\alpha+L_\alpha)\oplus 0_\alpha]-[L_\alpha\ominus L_\alpha]$$

which is also obtained from (5.13). The chain $[L_\alpha\ominus L_\alpha]$ bounds, by (10.9). Therefore, we have

$$K_\alpha\oplus L_\alpha\sim(K_\alpha+L_\alpha)\oplus 0_\alpha\sim K_\alpha+L_\alpha, \qquad (\Psi/\Phi),$$

by (11.17). As a corollary to (11.18), we can write

$$(11.19) \qquad [K_\alpha]+[L_\alpha]=[K_\alpha+L_\alpha].$$

The *product* of two homology classes will be defined by the formula

$$(11.20) \qquad [K][L] = [KL].$$

To justify the definition, we must, of course, prove the following:

$$(11.21) \qquad K \sim \overline{K}, L \sim \overline{L} \rightarrow KL \sim \overline{KL}, \qquad (\Psi/\Phi),$$

where all the chains involved are cycles of Ψ, modulo Φ. We begin with a lemma. Let K_α and L_β be cycles of Ψ, modulo Φ, and let M_γ be an arbitrary cycle of Γ, modulo Φ. Then we have

$$(11.22) \qquad K_\alpha \sim L_\beta \rightarrow \begin{cases} K_\alpha M_\gamma \sim L_\beta M_\gamma, \\ M_\gamma K_\alpha \sim M_\gamma L_\beta, \end{cases} \qquad (\Psi/\Phi).$$

PROOF. The homology $K_\alpha \sim L_\beta$ implies a relation of the form (11.2), which can be written in the alternative form

$$K_\alpha E_\beta E_\mu - E_\alpha L_\beta E_\mu \approx 0, \qquad (\Psi/\Phi),$$

by using the product notation for a refinement. Therefore, since M_γ is a cycle, modulo Φ, we can write

$$(K_\alpha E_\beta E_\mu - E_\alpha L_\beta E_\mu) M_\gamma \approx 0, \qquad (\Psi/\Phi),$$

from which we can infer

$$K_\alpha E_\beta E_\mu M_\gamma \sim E_\alpha L_\beta E_\mu M_\gamma, \qquad (\Psi/\Phi).$$

Thus, by refinement and permutation, we have

$$\begin{aligned} K_\alpha M_\gamma \sim K_\alpha M_\gamma E_\beta E_\mu \sim K_\alpha E_\beta E_\mu M_\gamma \sim E_\alpha L_\beta E_\mu M_\gamma \\ \sim L_\beta M_\gamma E_\alpha E_\mu \sim L_\beta M_\gamma, \end{aligned} \qquad (\Psi/\Phi).$$

The second half of the lemma is proved in a similar manner. The desired theorem (11.21) is an immediate consequence of the lemma, since we can now write $KL \sim K\overline{L} \sim \overline{K}\overline{L}$.

Class multiplication is *associative*, since chain multiplication is associative. We shall prove that it is *two-way distributive* with respect to chain addition:

$$(11.23) \quad K(L \oplus M) \sim KL \oplus KM, \quad (L \oplus M)K \sim LK \oplus MK, \quad (\Psi/\Phi).$$

PROOF. We have the identity

$$K_\alpha(L_\beta \oplus M_\gamma) = K_\alpha L_\beta E_\gamma + K_\alpha E_\beta M_\gamma.$$

Now, by (11.18), the expression on the right is homologous in Ψ, modulo Φ, to

$$K_\alpha L_\beta E_\gamma \oplus K_\alpha E_\beta M_\gamma,$$

which last is homologous, by (11.14), to

$$K_\alpha L_\beta \oplus K_\alpha M_\gamma,$$

since we have $K_\alpha L_\beta \sim K_\alpha L_\beta E_\gamma$ and $K_\alpha M_\gamma \sim K_\alpha M_\gamma E_\beta \sim K_\alpha E_\beta M_\gamma$.

In view of the properties of class addition and class multiplication, it is clear that the homology classes form an ordinary (noncommutative) ring. We shall call the ring the *homology ring* $\mathcal{H}(\Psi/\Phi)$ of the ideal Ψ, modulo the ideal Φ.

The ρth *homology group* $\mathcal{H}^\rho(\Psi/\Phi)$ of Ψ, modulo Φ, will be the additive group formed by the ρth constituents $[K]^\rho$ of the homology classes $[K]$. In view of (3.5) and (11.19), we shall have

$$[K] = [K]^0 + [K]^1 + \cdots + [K]^m, \tag{11.24}$$

and, by (3.7),

$$\begin{aligned} [K]^{\rho\rho} &= [K]^\rho, \\ [K]^{\rho\sigma} &= [0_\epsilon] \qquad (\rho \neq \sigma). \end{aligned} \tag{11.25}$$

In other words, the additive group formed by the members of the homology ring $\mathcal{H}(\Psi/\Phi)$, will be the (finite) direct sum of the homology groups $\mathcal{H}^\rho(\Psi/\Phi)$, $\rho = 0, 1, 2, \cdots$. In general, the structure of the homology ring $\mathcal{H}(\Psi/\Phi)$ will not be fully determined by the structure of the homology groups $\mathcal{H}^\rho(\Psi/\Phi)$. In other words, the ring $\mathcal{H}(\Psi/\Phi)$ will be a more powerful invariant than the combined groups $\mathcal{H}^\rho(\Psi/\Phi)$. Two homology rings $\mathcal{H}$ and $\mathcal{G}$ will be said to be *completely isomorphic* if, and only if, the members $[K]$ of $\mathcal{H}$ are paired in an isomorphism with the members $[L]$ of $\mathcal{G}$ in such a manner that the ρth constituent $[K]^\rho$ of $[K]$ is always paired with the ρth constituent $[L]^\rho$ of the corresponding $[L]$, $\rho = 0, 1, 2, \cdots$. The homology rings $\mathcal{H}$ and $\mathcal{G}$ will be regarded as equivalent if, and only if, they are *completely* isomorphic. In other words, a homology ring $\mathcal{H}$ will be treated as a ring with the operators ρ.

12. **The homology groups and ring of an ideal.** Let Γ be a grating, and let Ω be the ideal made up of the zeros 0_α of Γ. By the *homology ring* $\mathcal{H}(\Phi)$ of an ideal Φ of Γ, we shall mean the homology ring of Φ, modulo the ideal Ω,

$$\mathcal{H}(\Phi) = \mathcal{H}(\Phi/\Omega). \tag{12.1}$$

Similarly, by the *homology groups* $\mathcal{H}^\rho(\Phi)$ of Φ, we shall mean the homology groups

$$\mathcal{H}^\rho(\Phi) = \mathcal{H}^\rho(\Phi/\Omega). \tag{12.2}$$

The structure of the homology ring of an ideal Φ is simpler than that of a general homology ring $\mathcal{H}(\Psi/\Phi)$, as will be seen by the following analysis.

[12.1] THEOREM. *Every absolute* 0-*cycle* K *of a grating* Γ *is a multiple of a refinor,*

$$K' = 0 \to \text{some } K = \lambda E \qquad (\rho(K) = 0). \tag{12.3}$$

Every absolute ρ-cycle K of Γ, $\rho > 0$, is a boundary,

$$K' = 0 \to \text{some } K = R' \qquad (\rho(K) > 0). \tag{12.4}$$

PROOF. We prove the theorem by induction on the degree m of the cycle K. The case $m=0$ is trivial, since the only chains of degree 0 are the 0-cycles λE_ϵ, and since the only ρ-cycle of degree 0, $\rho > 0$, is the chain 0_ϵ which is its own boundary. The case $m > 0$ is treated by expressing the type α of K in the form $\alpha = a\beta$, where the degree of the type β is $m-1$. The ρ-cycle K can be written in the form

$$K = aL + bM + cN, \tag{12.5}$$

where L and N are ρ-chains of β, and where M is a $(\rho-1)$-chain of β. Therefore, the boundary of K is of the form

$$K' = aL' + b(L - M' - N) + cN' = 0, \tag{12.6}$$

from which we infer

$$L' = 0, \qquad L - M' - N = 0, \qquad N' = 0. \tag{12.7}$$

Let us first consider the case in which the rank of the cycle K is 0. In this case, the term bM in (12.5) is missing, and (12.7) reduces to

$$L' = 0, \qquad L - N = 0, \qquad N' = 0.$$

We conclude that L and N are identical absolute 0-cycles of rank $m-1$. Thus, by the hypothesis of the induction, we can assume

$$L = N = \lambda E_\beta,$$

whence,

$$K = aL + cN = \lambda(a + c)E_\beta = \lambda E_\alpha,$$

as required.

Finally, we consider the case in which the rank of K is greater than 0. According to (12.7), the chain L must be an absolute ρ-cycle of degree $m-1$. Therefore, by the hypothesis of the induction, we can express L as a boundary, $L = P'$. Moreover, again by (12.7), we can write $N = L - M' = P' - M'$. By eliminating L and N from (12.5), we obtain

$$K = aP' + bM + c(P' - M') = [(a + c)P - cM]',$$

which is of the desired form $K = R'$.

[12.2] COROLLARY. *The homology ring of the maximal ideal* Γ *of a grating* Γ *is made up of homology classes of rank* 0 *and is isomorphic with the ring of coefficients* $[\lambda]$.

PROOF. We have

$$K \sim K^0 + K^1 + \cdots + K^m \sim \lambda E_\epsilon \qquad (K^\rho \sim 0, \rho > 0),$$

so that each homology class is of the form $[\lambda E_\epsilon]$. Thus, to complete the proof, we have only to show that the classes $[\lambda E_\epsilon]$ and $[\mu E_\epsilon]$ are distinct whenever λ and μ are distinct. Now, if the classes were not distinct, we would have $[(\lambda - \mu)E_\epsilon] = [0_\epsilon]$, $\lambda \neq \mu$, which would imply that some refinement $(\lambda - \mu)E_\alpha$ of $(\lambda - \mu)E_\epsilon$ was a boundary. But the only chains of rank 0 which can bound are the zeros, since there are no chains of negative rank.

[12.3] COROLLARY. *Let* Φ *be any proper ideal of a grating* Γ, $\Phi \neq \Gamma$. *Then the product* KL *of two absolute cycles* K *and* L *of* Φ *always bounds in* Φ.

Proof. The chain K is an absolute cycle of Φ; therefore, *a fortiori*, it is an absolute cycle of Γ. Suppose we resolve the cycle K into its constituents,

$$K = K^0 + K^1 + \cdots + K^m.$$

Then, according to the theorem, the 0th constituent K^0 must be a multiple of a refinor, $K^0 = \lambda E$, while the other constituents K^ρ must be boundaries. Moreover, the coefficient λ of the 0th constituent $K^0 = \lambda E$ must vanish, otherwise λE would be a chain of Φ; whence all chains of Γ would be chains of Φ, since they would all be dominated by λE; whence Φ would not be a proper ideal of Γ, as is being assumed. We conclude that the chain K is a boundary, $K = R'$. We further conclude that the product KL is of the form $KL = R'L = (RL)' - R^*L' = (RL)'$, since the boundary L' of the absolute cycle L is a zero. To complete the argument, we have only to observe that the chain RL belongs to the ideal Φ, since L belongs to Φ. Since the product KL bounds the chain RL of Φ, the theorem is proved.

The homology ring of an ideal Φ is the analogue of the classical intersection ring of a region of the space Ξ^n of n real variables.

13. **Invariants of topological spaces.** We shall now prove two obvious lemmas which will have significant topological applications.

[13.1] LEMMA. *Let* (Γ, f, X) *be a representation of a grating* Γ *on a point set* X. *Then every subset* X_i *of* X *determines an ideal* Φ_i *of* Γ *consisting of all chains* K *of* Γ *such that their loci are made up of points of* X_i, $|K| \subseteq X_i$.

PROOF. The set Φ_i is an ideal of Γ if it fulfills Conditions (i), (ii), and (iii) at the beginning of §9. We verify, by inspection, that the set Φ_i fulfills Conditions (i) and (ii). It also fulfills Condition (iii), since $\|K\| \subseteq \|L\|$ implies $|K| \subset |L|$, by (8.1); whence $|L| \in X_s$ and $\|K\| \subseteq \|L\|$ together imply $|K| \subset |L| \in X_s$.

According to the lemma, there is an invariant ideal Φ_0 of the representation (Γ, f, X) consisting of all chains of Γ with empty loci. Thus, the homology rings $\mathcal{H}(\Gamma/\Phi_0)$ and $\mathcal{H}(\Phi_0)$, and the corresponding homology groups $\mathcal{H}^p(\Gamma/\Phi_0)$ and $\mathcal{H}^p(\Phi_0)$, are invariants of the representation.

[13.2] LEMMA. *Let* (Γ, f, X) *be a continuous representation of a grating* Γ *on a topological space* X. *Then there exists an ideal* Ψ_c *of* Γ *consisting of all chains* K *of* Γ *such that their loci are compact.*[3]

PROOF. The locus of the zero 0_ϵ is the empty set, which is compact; therefore, Condition (i) is satisfied. The union of two closed, compact sets is closed, compact; therefore, Condition (ii) is satisfied. Every closed subset of a closed, compact set is closed, compact; therefore, Condition (iii) is satisfied, since $\|K\| \subseteq \|L\|$ implies $|K| \subset |L|$.

According to the lemmas, there are two significant ideals of a continuous representation (Γ, f, X): the ideal Φ_0 consisting of the chains of Γ with empty loci, and the ideal Ψ_c consisting of the chains with compact loci. We can therefore obtain the following invariant rings of the representation: $\mathcal{H}(\Gamma/\Psi_c)$, $\mathcal{H}(\Gamma/\Phi_0)$, $\mathcal{H}(\Psi_c/\Phi_0)$, $\mathcal{H}(\Psi_c)$, $\mathcal{H}(\Phi_0)$. Among these invariants, the most significant one is the ring $\mathcal{H}(\Psi_c/\Phi_0)$. We shall call this ring the *homology ring* of the representation. Moreover, we shall call the associated groups $\mathcal{H}^p(\Psi_c/\Phi_0)$ the *homology groups* of the representation.

Given a topological space X, there are certain grating representations (Γ, f, X) on X which are themselves topological invariants of the space. The invariant groups and rings of these representations are, of course, invariants of the space. The representation described

[3] By a *compact* set, we shall mean a set satisfying the *complete* Heine-Borel-Lebesgue condition. In the terminology of Alexandroff and Hopf, such a set would be said to be *bicompact.*

in Example A of §7 is clearly a topological invariant of the space X. Other significant invariant representations are the following:

Example B. Let X be a topological space, and let $[\gamma]$ be the set of all closed subsets of X. We treat each member γ_i of $[\gamma]$ as the symbol for a cut consisting of three abstract elements a_i, b_i, and c_i. The cuts γ_i form a grating $\Gamma=[\gamma]$. Moreover, there is a continuous representation (Γ, f, X) of the grating Γ on the space X, such that the determining function of the representation is the function $f(\gamma, x)$ which takes on the values -1, 0, or 1 according as the point x is interior to, on the frontier of, or exterior to the closed set γ.

Example C. Let X be a topological space, and let $[\gamma]$ be the set of all ordered pairs $\gamma_i=(A_i, C_i)$ of disjoint open subsets A_i and C_i of X. We treat each member γ_i of $[\gamma]$ as the symbol for a cut with the elements a_i, b_i, and c_i. The cuts γ_i form a grating $\Gamma=[\gamma]$. Again, there is a continuous representation (Γ, f, X) of Γ on X, such that the determining function $f(\gamma, x)$ is the one which takes on the value -1 when x is a point of A, the value 1 when x is a point of C, and the value 0 when x is a point of the complement of $A\cup C$.

In a continuation of this paper, which will deal specifically with the the applications of grating theory to hausdorff spaces, we shall prove the following theorem:

Given a locally compact hausdorff space X, the homology ring $\Gamma(\Psi_c/\Phi_0)$ of the continuous representation (Γ, f, X) described in Example A of §7 is completely isomorphic with the homology rings of the representations described in Examples B and C. Moreover, if the space X is an (open or closed) manifold, the homology ring $\Gamma(\Psi_c/\Phi_0)$ is completely isomorphic with the classical intersection ring of the manifold, provided the classical cycles of dimensionality k are interpreted as of rank $\rho=n-k$.

If X is a space of unrestricted generality, the homology ring of the representation in Example A need not be completely isomorphic with the homology rings of the representations in Examples B and C.

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