A PROPERTY OF DERIVATIVES

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If a real function defined over a closed interval $[a, b]$ is differentiable at each point of the interval, it is well known that its derivative possesses the Darboux property: if $f'(c) < \xi < f'(d)$, then there is a point e between c and d with $f'(e) = \xi$.

Now let α, β, with $\alpha < \beta$, be any two fixed reals, and consider the set $E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}$. It is easily seen, as a consequence of the Darboux property, that any such set $E(\alpha, \beta)$ must contain a continuum of points, unless it is empty. The question of the measure of $E(\alpha, \beta)$ does not seem to be covered in the literature, except in the case in which the given interval is either $(-\infty, \beta)$ or $(\alpha, +\infty)$. We prove that any such set $E(\alpha, \beta)$ is either empty or of positive measure.

We remark that this result cannot be deduced from the Darboux property alone; Lebesgue exhibited a function[1] which possesses that property without satisfying the measure condition. Another example is the following. Let C be the Cantor closed nondense set of measure zero and power c in the unit interval, and let $\{T_n\}$ $(n = 1, 2, 3, \cdots)$ be a sequence of linear transformations such that the sets $T_n(C)$ are disjoint, and such that any sub-interval of $[0, 1]$ contains some $T_n(C)$. We take T_1 to be the identity. Let the function $g(x)$ be defined on C in such a way as to assume all values from zero to one inclusive; on $T_n(C)$ let $g(x) = g(T_n^{-1}(x))$. On all remaining points of the unit interval set $g(x) = 0$. It is clear that this function g possesses the Darboux property, but that the set $E\{x/1/2 < g(x) < 1\}$ will be nonvoid and of measure zero.

THEOREM. *If $f(x)$ is real and everywhere differentiable in the closed interval $[a, b]$, then for any two reals α, β $(\alpha < \beta)$, the set*

$$E(\alpha, \beta) = E\{x/\alpha < f'(x) < \beta\}$$

is empty or of positive measure.

PROOF. We start with the following known result:[2] if a continuous function $f(x)$ is differentiable in the interval $[a, b]$, with the possible exception of a denumerable set of points x, and if $f'(x)$ is non-negative almost everywhere, then $f(x)$ is nondecreasing. It follows that if $f'(x)$ exists for all x in $[a, b]$, and $f'(x) \geqq \lambda$ [or $f'(x) \leqq \mu$] for almost all x,

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[1] Lebesgue, *Leçons sur l'intégration*, 2d ed., Paris, 1928, p. 97.

[2] Saks, *Théorie de l'intégrale*, Warsaw, 1933, p. 141.

then $f'(x) \geqq \lambda$ $[f'(x) \leqq \mu]$ for all x in $[a, b]$ (by considering the functions $f-\lambda x$, $f-\mu x$). In other words, if a function is differentiable everywhere, then the least upper and greatest lower bounds of its derivative are not changed if sets of measure zero are neglected in computing them: this is the theorem which results from our formulation if either $\alpha = -\infty$ or $\beta = +\infty$. We shall make use of this remark in the following argument.

Consider α, β arbitrary and finite: write E for $E(\alpha, \beta)$. Suppose E not void, and $mE=0$; we shall show that a contradiction results. We write $E_\alpha = E\{x/f'(x) \leqq \alpha\}$, $E_\beta = E\{x/f'(x) \geqq \beta\}$, so that the interval $[a, b] = E + E_\alpha + E_\beta$, the three sets being disjoint.

First, $E \subset E_\alpha' \cdot E_\beta'$. To prove this relation, assume the contrary and let x_0 belong to E but not to E_α'; then x_0 is in the interior of some interval U, $U \cdot E_\alpha$ void. But then in the interval U the set of points where $f' < \beta$ is not empty, since it contains x_0, but is of measure zero, since it is a subset of E. This contradicts the special case of our theorem which was referred to in our preliminary remark. Thus $E \subset E_\alpha'$, and by a similar argument, $E \subset E_\beta'$ also.

The derivative $f'(x)$ belongs to Baire's first class, and hence, by a known theorem,[3] if A is any subset of $[a, b]$, and f' be considered for the moment on the domain A alone, its points of discontinuity must form a set of the first category relative to A. Take for the subset A the closure of our set E: this set, $\overline{E}$, being closed, is of second category relative to itself, and hence we shall have a contradiction if we show that $f'(x)$ is everywhere discontinuous considered on the domain $\overline{E}$. This may be seen as follows: let x_0 be a point, first, of E itself. Since $E \subset E_\alpha' \cdot E_\beta'$, we have for any interval I which contains x_0

$$\sup_{x \in I} f'(x) \geqq \beta, \qquad \inf_{x \in I} f'(x) \leqq \alpha.$$

Because f' possesses the Darboux property, we infer

$$\sup_{x \in I \cdot E} f'(x) = \beta, \qquad \inf_{x \in I \cdot E} f'(x) = \alpha,$$

so that we have now shown that, considered on domain E, and so a fortiori on domain $\overline{E}$, the function f' is discontinuous at each point of E. On domain $\overline{E}$ the saltus of f' at each point of E has been shown to be at least $\beta - \alpha$; the same, then, will be true at each point of $\overline{E}$. The function f' has now been shown to be everywhere discontinuous considered on domain $\overline{E}$; this completes the proof.

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[3] Kuratowsky, *Topologie*, Warsaw, 1933, p. 189.