SUBSERIES OF SERIES WHICH ARE NOT ABSOLUTELY CONVERGENT

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1. **Introduction.** It is the object of this note to give two theorems on series, of real or complex terms, which fail to converge absolutely. The second is a corollary of the first. They say, roughly, that each such series becomes or remains divergent after "nearly all" of its terms, suitably selected, are discarded or replaced by zeros.

THEOREM 1. *If* $a(1)+a(2)+a(3)+\cdots$ *is a series of real or complex terms which fails to converge absolutely, then there is an increasing sequence*

$$1 \leqq n_1 < n_2 < n_3 < \cdots \tag{1}$$

of integers such that $n_{k+1}-n_k \to \infty$ *as* $n \to \infty$ *and the series*

$$a(n_1) + a(n_2) + a(n_3) + \cdots \tag{2}$$

is divergent.

THEOREM 2. *If* $a(1)+a(2)+a(3)+\cdots$ *is a series of real or complex terms which fails to converge absolutely, then there is a sequence* $x_1, x_2, x_3, \cdots$ *of which each element is either* 0 *or* 1, *such that*

$$\lim_{n\to\infty} (x_1 + x_2 + \cdots + x_n)/n = 0 \tag{3}$$

and the series

$$a(1)x_1 + a(2)x_2 + a(3)x_3 + \cdots \tag{4}$$

is divergent.

2. **Series of non-negative terms.** In this section, we obtain the conclusions of Theorems 1 and 2 for the case in which $a(1)+a(2)+\cdots$ is a series of real non-negative terms which fails to converge absolutely and accordingly $a(1)+a(2)+\cdots$ is a divergent series of real non-negative terms. Choose integers

$$1 = \alpha_2 < \beta_2 < \alpha_3 < \beta_3 < \alpha_4 < \beta_4 < \cdots \tag{5}$$

such that, for each $p=2, 3, 4, \cdots$,

$$\sum_{k=\alpha_p}^{\beta_p-1} a(k) > p, \tag{6}$$

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$$\alpha_{p+1} - \beta_p > p, \tag{7}$$

and

$$\beta_p - \alpha_p = p\gamma_p \tag{8}$$

where $\gamma_2, \gamma_3, \gamma_4, \cdots$ are integers. For each index p,

$$\sum_{k=0}^{p-1} \sum_{j=0}^{\gamma_p-1} a(\alpha_p + k + jp) = \sum_{k=\alpha_p}^{\beta_p-1} a(k) > p \tag{9}$$

and hence there must be an index k_p such that $0 \leqq k_p < p$ and

$$\sum_{j=0}^{\gamma_p-1} a(\alpha_p + k_p + jp) > 1. \tag{10}$$

Let $x_n = 0$ when $\alpha_p \leqq n < \alpha_{p+1}$ except that $x_n = 1$ when

$$n = \alpha_p + k_p + jp, \qquad j = 0, 1, \cdots, \gamma_p - 1. \tag{11}$$

The series (4) is divergent because

$$\sum_{n=\alpha_p}^{\beta_p-1} a(n)x_n = \sum_{j=0}^{\gamma_p-1} a(\alpha_p + k_p + jp) > 1. \tag{12}$$

Let $n_1, n_2, n_3, \cdots$ denote in increasing order the values of n for which $x_n = 1$. The series (2) then becomes the series obtained by omitting zero terms from (4) and hence (2) is divergent. Moreover the sequence $n_1, n_2, n_3, \cdots$ is so constructed that $n_{k+1} - n_k \to \infty$ as $n \to \infty$, and therefore (3) holds. This gives the conclusions of the theorems for the case in which the series have real non-negative terms.

3. **Series of complex terms.** Let $a(1) + a(2) + \cdots$ be a series of complex terms for which $\sum |a(n)| = \infty$. Let $a(n) = b(n) + ic(n)$ where $b(n)$ and $c(n)$ are real. Then at least one of

$$\sum_{k=1}^{\infty} | b(k) | = \infty, \qquad \sum_{k=1}^{\infty} | c(k) | = \infty \tag{13}$$

holds. If the first holds, let $d(k) = b(k)$; otherwise, let $d_k(t) = c_k(t)$. Let

$$d^+(k) = 2^{-1}[\, | d(k) | + d(k)], \quad d^-(k) = 2^{-1}[\, | d(k) | - d(k)] \tag{14}$$

so that $d^+(k) \geqq 0$, $d^-(k) \geqq 0$ and $d(k) = d^+(k) - d^-(k)$. Then at least one of

$$\sum_{k=1}^{\infty} d^+(k) = \infty, \qquad \sum_{k=1}^{\infty} d^-(k) = \infty \tag{15}$$

holds. If the first holds, let $e(k)=d^+(k)$; otherwise, let $e(k)=d^-(k)$. Then $\sum e(k)$ is a divergent series of real non-negative terms. Therefore, as was proved in §2, there is an increasing sequence n_k such that $n_{p+1}-n_p\to\infty$ and the series

$$e(n_1)+e(n_2)+e(n_3)+\cdots \tag{16}$$

is divergent. Let

$$e(n_1')+e(n_2')+e(n_3')+\cdots \tag{17}$$

be the subseries of (16) obtained from (16) by omitting all zero terms. Then $n_{p+1}'-n_p'\to\infty$ and (17) is divergent. Since $d^+(k)=0$ when $d^-(k)\neq 0$, and $d^-(k)=0$ when $d^+(k)\neq 0$, it follows that the series

$$d(n_1')+d(n_2')+d(n_3')+\cdots$$

is divergent. Hence at least one of $\sum b(n_k')$ and $\sum c(n_k')$ is divergent and therefore $\sum a(n_k')$ is divergent. This completes the proof of Theorem 1 and hence also that of Theorem 2.

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