ON AREOLAR MONOGENIC FUNCTIONS

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Let $f(z)=u(x, y)+iv(x, y)$, $z=x+iy$, be a complex-valued function defined in the unit circle D: $|z|<1$. $f(z)$ is said to be *areolar monogenic* in D if and only if $u(x, y)$ and $v(x, y)$ (and hence $f(z)$) have continuous partial derivatives of the second order such that

$$u_{xy} = -2^{-1}(v_{xx} - v_{yy}), \qquad v_{xy} = 2^{-1}(u_{xx} - u_{yy}) \tag{1}$$

hold in D [3].[1] It is known that an areolar monogenic function has partial derivatives of all orders [3].

Whereas (1) is a differential characterization of areolar monogenic functions, it is the integral characterization contained in the following theorem that forms the basis for this note.

THEOREM A [3]. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that there exist a function $g(z)$, analytic in D, such that*

$$g(z) = \frac{1}{\pi r^2 i}\int_{C(z;r)} f(\zeta)d\zeta, \qquad \zeta = \xi + i\eta, \tag{2}$$

holds for all circles $C(z; r)$, with center z and radius r, in D.

It should be noted that a symbol once introduced holds its meaning throughout the paper.

If $f(z)$ is continuous in D, then the right-hand member of (2) is a function of z and r, defined for z in the circle D_r: $|z|<1-r$, and for all r such that $0<r<1$. Now if the definition

$$G_r(z) \equiv \frac{1}{\pi r^2 i}\int_{C(z;r)} f(\zeta)d\zeta, \qquad 0 < r < 1,\ |z| < 1 - r, \tag{3}$$

is made, then the following is an extension of Theorem A.

THEOREM 1. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that $G_r(z)$ be analytic in D_r, for all r, $0<r<1$.*

Necessity. This is precisely the necessity part of Theorem A.

Sufficiency. Suppose that in addition $f(z)$ has continuous partial

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[1] Numbers in brackers refer to the bibliography at the end of the paper.

derivatives of the first order in D. Then from (3) and Green's lemma it follows that

$$G_r(z) = \frac{1}{\pi r^2} \iint_{D(z;r)} G(\zeta) d\xi d\eta, \tag{4}$$

where

$$G(z) \equiv U(x, y) + iV(x, y) \equiv (u_x - v_y) + i(u_y + v_x), \tag{5}$$

and where $D(z; r)$ is the closed circular disc with center at z and radius r. Since $f(z)$ has continuous partial derivatives of the first order in D, it follows from (4) and (5) that $G_r(z) \rightarrow G(z)$, as $r \rightarrow 0$, on each closed subset of D. Hence, since $G_r(z)$ is analytic in D_r, for all r, it follows that $G(z)$ is analytic in D. Therefore $U(x, y)$ and $V(x, y)$ are conjugate harmonic functions such that (4) and the Gauss mean-value theorem for harmonic functions yield

$$G_r(z) = \frac{1}{\pi r^2} \iint_{D(z;r)} G(\zeta) d\xi d\eta = G(z)$$

for all z in D_r, $0 < r < 1$.

It now follows from Theorem A that $f(z)$ is areolar monogenic in D.

Now if $f(z)$ is merely continuous in D, then for sufficiently small ρ, the mean-value function

$$A(f; z; \rho) \equiv \frac{1}{\pi \rho^2} \iint_{D(z;\rho)} f(\zeta) d\xi d\eta \tag{6}$$

satisfies the hypotheses of this lemma in the circle D_ρ, for $0 < \rho < 1$; moreover, $A(f; z; \rho)$ has continuous partial derivatives of the first order in D_ρ [1]. Therefore by the preceding part of this proof, it follows that for all sufficiently small ρ, the function

$$\frac{1}{\pi r^2 i} \int_{C(z;r)} A(f; \zeta; \rho) d\zeta, \qquad 0 < r + \rho < 1,$$

is analytic in D_{r+p} and independent of r. But $A(f; z; \rho) \rightarrow f(z)$ as $\rho \rightarrow 0$, on each closed subset of D [1]. Hence it follows that for $0 < r < 1$, the right-hand member of (2) is analytic and independent of r in D_r. Hence, by Theorem A, $f(z)$ is areolar monogenic in D.

Corollary 1. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that $A(f; z; \rho)$ be areolar monogenic in D_ρ, for all ρ, $0 < \rho < 1$.*

Corollary 2. *If $f(z)$ is areolar monogenic in D, then $f(z)$ has the following mean-value property:*

$$\int_{C(z;r)} f(\zeta)d\zeta = \int_{C(z;r)} A(f;\zeta;\rho)d\zeta$$

for each $C(z;r)$ *in* D_ρ.

Proofs of the corollaries are contained in the proof of Theorem 1.

The equation (2) and recent results on the polygonal mean-values of harmonic polynomials [2] suggest the analogue of (2) wherein $C(z;r)$ is replaced by a regular n-gon $p_n(z;r;\phi)$, $n \geqq 3$. Let $P_n(z;r;\phi)$ denote the closed, finite region bounded by the regular n-gon $p_n(z;r;\phi)$ whose center is at z and whose inscribed circle has radius r; ϕ denotes the angle from R to N, $-\pi/n \leqq \phi < \pi/n$, where R is the ray extending horizontally to the right from z and N is the exterior normal at the point where R emerges from the polygon.

Here n denotes a fixed, positive integer, $n \geqq 3$, and ϕ always denotes an (arbitrary) angle, $-\pi/n \leqq \phi < \pi/n$. For brevity, $p_n(z;r;\phi)$ will be denoted by $p(z;r)$ and $P_n(z;r;\phi)$ will be denoted by $P(z;r)$; $|P|$ will denote the area of $P(z;r)$.

The analogue of (2) referred to above is

$$(7) \qquad F_{r,\phi}(z) \equiv \frac{1}{|P|\, i} \int_{p(z;r)} f(\zeta)d\zeta,$$

which is defined on an open subset $D_{r,\phi}$ of D, for sufficiently small r.

The following result is comparable to the preceding theorem.

THEOREM 2. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $f(z)$ be areolar monogenic in D is that $F_{r,\phi}(z)$ be analytic in $D_{r,\phi}$ for each pair r, ϕ.*

Necessity. If $f(z)$ is areolar monogenic in D, then $f(z)$ has continuous partial derivatives (of all orders) in D [3]. Hence (7) and Green's lemma yield the following representation for $F_{r,\phi}(z)$:

$$(8) \qquad F_{r,\phi}(z) = \frac{1}{|P|} \iint_{P(z;r)} G(\zeta)d\xi d\eta,$$

where, by (1) and (5), $G(z)$ is analytic in D. Now (8) shows that $F_{r,\phi}(z)$ is an integral mean of $G(z)$, so that $F_{r,\phi}(z)$ is analytic wherever defined, that is, in $D_{r,\phi}$.

Sufficiency. First suppose that $f(z)$ has continuous partial derivatives of the first order in D. Then it follows from (7) and Green's lemma that $F_{r,\phi}(z)$ can be written in the form (8). It now follows from (8) that $F_{r,\phi}(z) \rightarrow G(z)$, as $r \rightarrow 0$, on each closed subset of D. Since $F_{r,\phi}(z)$ is analytic in $D_{r,\phi}$, for all sufficiently small r, it follows that

$G(z)$ is analytic in D. Hence, as in the proof of Theorem 1, it follows that $f(z)$ is areolar monogenic in D.

The requirement that $f(z)$ have continuous partial derivatives of the first order in D may be removed as in the proof of Theorem 1. This completes the proof.

It should be noted that if $f(z)$ is an arbitrary areolar monogenic function (hence, if $G(z)$ is an arbitrary analytic function) in D, then $F_{r,\phi}(z)$ is analytic in $D_{r,\phi}$, though not necessarily independent of r. Indeed, if $F_{r,\phi}(z)$ is to be both analytic and independent of r, in $D_{r,\phi}$, then the following result holds.

THEOREM 3. *If $f(z)$ is continuous in D, then a necessary and sufficient condition that $F_{r,\phi}(z)$ be both analytic and independent of r in $D_{r,\phi}$, for fixed ϕ, is that $f(z)$ be areolar monogenic in D, with the representation*

$$f(z) \equiv 2^{-1}\sum_{0}^{n-1} c_k(\bar{z}z^k - z^{k-1}) + \Psi_y + i\Psi_x, \tag{9}$$

where $\bar{z} = x - iy$, where the c_k are arbitrary complex constants, and where $\Psi(x, y)$ is an arbitrary function harmonic in D.

To prove Theorem 3, the following lemma is needed.

LEMMA. *If $f(z)$ is areolar monogenic in D, then a necessary and sufficient condition that $G(z)$ be a polynomial in z of degree at most $(n-1)$ is that $f(z)$ have the representation* (9).

Necessity. Let $G(z)$ have the representation

$$G(z) = \sum_{0}^{n-1} c_k z^k. \tag{10}$$

Now Haskell has shown [3] that for areolar monogenic $f(z)$ there exist real functions $\mu(x, y)$, $\nu(x, y)$, $\Psi(x, y)$, with $\Psi(x, y)$ harmonic in D, such that

$$f(z) \equiv (\mu_x + \nu_y) + i(\nu_x - \mu_y) + \Psi_y + i\Psi_x, \tag{11}$$

where

$$\mu + i\nu \equiv -\frac{1}{2\pi}\iint_D \log\frac{1}{|z-\zeta|}G(\zeta)d\xi d\eta. \tag{12}$$

If the substitutions $\zeta = re^{i\theta}$, $z = \rho e^{i\psi}$ are made in (10) and (12), then (12) yields

$$
\begin{aligned}
\mu + i\nu = & -\frac{1}{2\pi}\int_0^\rho \int_0^{2\pi} \left[\log\frac{1}{\rho} + \sum_1^\infty \frac{r^k}{k\rho^k}\cos k(\theta-\psi)\right] \\
& \cdot\left[\sum_0^{n-1} c_k r^k e^{ik\theta}\right] r dr d\theta \\
& -\frac{1}{2\pi}\int_\rho^1 \int_0^{2\pi} \left[\log\frac{1}{r} + \sum_1^\infty \frac{\rho^k}{kr^k}\cos k(\theta-\psi)\right] \\
& \cdot\left[\sum_0^{n-1} c_k r^k e^{ik\theta}\right] r dr d\theta \\
= & \frac{1}{4}\left[\sum_0^{n-1} \frac{c_k}{k+1} \bar{z}z^{k+1} - \sum_1^{n-1} \frac{c_k}{k} z^k - c_0\right].
\end{aligned} \tag{13}
$$

The representation (9) for $f(z)$ now follows from (11) and (13).

Sufficiency. If $f(z)$ is given by (9), then a computation shows that $G(z)$, given by (5), has the form (10).

PROOF OF THEOREM 3. If $f(z)$ is areolar monogenic in D, with representation (9), then it follows from the lemma that $G(z)$ has the form (10), such that $U(x, y)$ and $V(x, y)$ are harmonic polynomials of degree at most $(n-1)$. It is known that such harmonic polynomials satisfy

$$
\begin{aligned}
U(x, y) \equiv U(z) &= \frac{1}{|P|}\iint_{P(z;r)} U(\zeta) d\xi d\eta, \\
V(x, y) \equiv V(z) &= \frac{1}{|P|}\iint_{P(z;r)} V(\zeta) d\xi d\eta,
\end{aligned} \tag{14}
$$

for each $P(z; r)$ in D [2]. From (5), (8) and (14), it follows that $F_{r,\phi}(z)$ is independent of r, ϕ. This proves the necessity part of the theorem.

On the other hand, if $F_{r,\phi}(z)$ is analytic and independent of r in $D_{r,\phi}$, then by the lemma, $f(z)$ is areolar monogenic in D. Moreover, it follows that the real and imaginary parts of $G(z)$ satisfy (14) and hence $U(x, y)$ and $V(x, y)$ have the representations implied by

$$
G(z) \equiv U + iV = \sum_0^{n-1} c_k z^k + c_n I(z_k \alpha), \qquad \alpha = e^{i\psi}, \tag{15}
$$

where the symbol "I" means "the imaginary part of" [2]. However, since $U(x, y)$ and $V(x, y)$ are conjugate harmonic functions, it follows that $c_n = 0$ in (15). Hence $G(z)$ is a polynomial of degree at most $(n-1)$; therefore, by the lemma, $f(z)$ is areolar monogenic in D with representation (9). The proof is now complete.

The author is indebted to the referee for the observation that Theorem 3 above is true for variable ϕ.

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