RECIPROCALS OF J-MATRICES

H. S. WALL

1. **Introduction.** We consider J-matrices

$$J = (j_{pq}), \qquad j_{pq} = 0 \quad \text{for} \quad |p-q| \geqq 2, \qquad j_{pp} = b_p,$$
$$j_{p+1,p} = j_{p,p+1} = -a_p \neq 0,$$

such that

$$(1.1) \quad I[J(x, \bar{x})] = \sum I(b_p)|x_p|^2 - \sum I(a_p)(x_p\bar{x}_{p+1} + \bar{x}_p x_{p+1}) \geqq 0$$

for all x_p for which the sums converge. These are the J-matrices associated with a positive definite J-fraction [4, 5, 1].[1] Let $X_p(z)$ and $Y_p(z)$ denote the solutions of the system of linear equations

$$(1.2) \quad -a_{p-1}x_{p-1} + (b_p + z)x_p - a_p x_{p+1} = 0, \; p = 1, 2, 3, \cdots; a_0 = 1,$$

under the initial conditions $x_0 = -1$, $x_1 = 0$ and $x_0 = 0$, $x_1 = 1$, respectively. We shall prove that when at least one of the series

$$(1.3) \qquad \sum_{p=1}^{\infty} |X_p(0)|^2, \qquad \sum_{p=1}^{\infty} |Y_p(0)|^2$$

diverges, then the matrix $J+zI$ has a unique bounded reciprocal for $I(z) > 0$, and that when both the series (1.3) converge then the matrix $J+zI$ has infinitely many different bounded reciprocals. This theorem was proved by Hellinger [2] for the case where the coefficients a_p and b_p are all real.

2. **Reciprocals of an arbitrary J-matrix.** The general right reciprocal of $J+zI$ is (ρ_{pq}) where $\rho_{1,q}$, $q = 1, 2, 3, \cdots$, are arbitrary functions of z, and [3, p. 116]

$$(2.1) \qquad \rho_{pq}(z) = \begin{cases} \rho_{1,q}(z)Y_p(z), & p = 1, 2, 3, \cdots, q; \\ \rho_{1,q}(z)Y_p(z) + X_q(z)Y_p(z) - X_p(z)Y_q(z), & \\ & p = q+1, q+2, q+3, \cdots. \end{cases}$$

We shall say that the *determinate case* or the *indeterminate case* holds for the J-matrix according as at least one of the series (1.3) diverges or both of these series converge, respectively. In the indeterminate case, both of the series

---

Received by the editors December 26, 1945.

[1] Numbers in brackets refer to the Bibliography at the end of the paper.

$$(2.2) \qquad \sum_{p=1}^{\infty} | X_p(z) |^2, \qquad \sum_{p=1}^{\infty} | Y_p(z) |^2$$

converge for every value of z [3, p. 120]. Hence if the functions $\rho_{1,q}(z)$ are chosen such that the series $\sum | \rho_{1,q}(z) |^2$ converges, it follows by (2.1) and Schwarz's inequality that the double series $\sum | \rho_{pq}(z) |^2$ converges and therefore the matrix $(\rho_{pq}(z))$ is bounded. If, in particular,

$$\rho_{1,q}(z) = Y_q(z)f(z) - X_q(z),$$

then

$$(2.3) \quad \rho_{pq}(z) = \begin{cases} Y_p(z)Y_q(z)\left(f(z) - \dfrac{X_q(z)}{Y_q(z)}\right), & p = 1, 2, \cdots, q; \\[2ex] Y_p(z)Y_q(z)\left(f(z) - \dfrac{X_p(z)}{Y_p(z)}\right), & p = q+1, q+2, \cdots, \end{cases}$$

so that the matrix (ρ_{pq}) is symmetric. If, for example, $f(z)$ is an entire function, then the matrix (ρ_{pq}) given by (2.3) is bounded in the indeterminate case for all values of z. Hence we have the following theorem.

THEOREM 2.1. *In the indeterminate case, the J-matrix $J+zI$ has infinitely many different reciprocals $(\rho_{pq}(z))$ which are bounded for all values of z.*

We have not used the condition (1.1), so that this theorem holds for arbitrary J-matrices.

3. **The determinate case.** We suppose now that (1.1) holds. Then

$$(3.1) \qquad \beta_p = I(b_p) \geqq 0, \qquad p = 1, 2, 3, \cdots,$$

and there exist constants g_p such that if $\alpha_p = I(a_p)$ then

$$(3.2) \quad \alpha_p^2 = \beta_p\beta_{p+1}(1 - g_{p-1})g_p, \qquad 0 \leqq g_{p-1} \leqq 1, \ p = 1, 2, 3, \cdots.$$

Conversely, if (3.1) and (3.2) hold, then (1.1) holds [5, p. 91].

For a fixed positive integer n, let $\xi_1, \xi_2, \cdots, \xi_n$ be arbitrary real numbers. Let (ρ_{pq}) be any right reciprocal of $J+zI$, so that, if we now take $a_0 = 0$,

$$(3.3) \quad \begin{aligned} - a_{p-1}\rho_{p-1,q} + (b_p + z)\rho_{p,q} - a_p\rho_{p+1,q} &= \delta_{p,q}, \\ p, q &= 1, 2, 3, \cdots, \end{aligned}$$

where $\delta_{p,q} = 0$ or 1 according as $p \neq q$ or $p = q$, respectively. On multiplying (3.3) by ξ_q and summing over q from 1 to n we obtain

$$-a_{p-1}\eta_{p-1} + (b_p + z)\eta_p - a_p\eta_{p+1} = \xi_p, \tag{3.4}$$

where

$$\eta_p = \sum_{q=1}^{n} \rho_{pq}\xi_q. \tag{3.5}$$

We now suppose (ρ_{pq}) is symmetric, so that (2.3) holds for some function $f(z)$. We note that

$$w = \frac{a_n\rho_{n,q}}{\rho_{n+1,q}}$$

is independent of q for $q=1, 2, 3, \cdots, n$:

$$w = a_n \frac{Y_n(z)f(z) - X_n(z)}{Y_{n+1}(z)f(z) - X_{n+1}(z)}, \quad f(z) = \frac{X_{n+1}(z)w - a_nX_n(z)}{Y_{n+1}(z)w - a_nY_n(z)}. \tag{3.6}$$

For a fixed z with $I(z)>0$, the transformation

$$t = \frac{X_{n+1}(z)w - a_nX_n(z)}{Y_{n+1}(z)w - a_nY_n(z)}$$

maps the half-plane $I(w) \geqq \beta_{n+1}g_n$ upon a circular region $K_n(z)$ (cf. [1]). Hence we see by (3.6) that the value of the function $f(z)$ is in $K_n(z)$ if and only if $I(w) \geqq \beta_{n+1}g_n$. If the latter inequality holds then [1, p. 261]

$$\beta_n + y - I\left(\frac{a_n^2}{w}\right) \geqq \beta_n g_{n-1} + y, \qquad \text{where} \quad y = I(z) > 0,$$

or

$$I\left(\frac{a_n^2}{w}\right) \leqq \beta_n(1 - g_{n-1}). \tag{3.7}$$

Now

$$a_n\rho_{n+1,q} = \frac{a_n^2}{w}\rho_{n,q}.$$

On multiplying this by ξ_q and summing over q from 1 to n, we get

$$a_n\eta_{n+1} = \frac{a_n^2}{w}\eta_n. \tag{3.8}$$

We now multiply (3.4) by $\bar{\eta}_p$, sum over p from 1 to n, and eliminate the quantity $a_n\eta_{n+1}\bar{\eta}_n$ by (3.8). This gives immediately the relation

$$\sum_{p=1}^{n}(b_p+z)\mid\eta_p\mid^2-\sum_{p=1}^{n-1}a_p(\eta_p\bar{\eta}_{p+1}+\bar{\eta}_p\eta_{p+1})=\frac{a_n^2}{w}\mid\eta_n\mid^2+\sum_{p=1}^{n}\xi_p\bar{\eta}_p.$$

If we consider only the imaginary part and make use of the inequality (3.7) and the relations (3.2) we then obtain (cf. [1, p. 258])

$$\begin{aligned}y\sum_{p=1}^{n}\mid\eta_p\mid^2+\sum_{p=1}^{n-1}\mid(\beta_p(1-g_{p-1}))^{1/2}\eta_p-(\beta_{p+1}g_p)^{1/2}\eta_{p+1}\mid^2\\+\sum_{p=1}^{n}\xi_pI(\eta_p)\leqq 0.\end{aligned}\tag{3.9}$$

Hence, in particular,

$$y\sum_{p=1}^{n}\mid\eta_p\mid^2+\sum_{p=1}^{n}\xi_pI(\eta_p)\leqq 0.\tag{3.10}$$

This holds under the assumption that the value of the function $f(z)$ is in the circular region $K_n(z)$.

Turning now to the quadratic form

$$R_n(\xi,\xi)=\sum_{p,q=1}^{n}\rho_{pq}(z)\xi_p\xi_q=\sum_{p=1}^{n}\xi_p\eta_p,$$

we have, by Schwarz's inequality and (3.10),

$$\begin{aligned}\mid R_n(\xi,\xi)\mid^2&=\left|\sum_{p=1}^{n}\xi_p\eta_p\right|^2\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\cdot y\sum_{p=1}^{n}\mid\eta_p\mid^2\\&\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\left(-\sum_{p=1}^{n}\xi_pI(\eta_p)\right)\\&=\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\cdot(-I[R_n(\xi,\xi)]).\end{aligned}$$

Therefore,

$$\mid R_n(\xi,\xi)\mid^2\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2\cdot\mid R_n(\xi,\xi)\mid,$$

or

$$\mid R_n(\xi,\xi)\mid\leqq\frac{1}{y}\sum_{p=1}^{n}\xi_p^2.\tag{3.11}$$

This holds for any particular values of n and z, $I(z)>0$, such that the value of $f(z)$ is in $K_n(z)$. Now [1, §3], $K_1(z)\supset K_2(z)\supset K_3(z)\supset \cdots$, and there is at least one function $f(z)$ which is analytic for $I(z)>0$ whose values are in *all* the circles $K_n(z)$. Hence we conclude that the following theorem is true.

THEOREM 3.1. *If* (1.1) *holds, then the matrix* $J+zI$ *has at least one reciprocal which is bounded for* $I(z)>0$.

We shall now prove the following theorem.

THEOREM 3.2. *If* (1.1) *holds, then, in the determinate case, the matrix* $J+zI$ *has just one reciprocal which is bounded for all* z *for which* $I(z)>0$.

PROOF. In the determinate case at least one of the series (2.2) diverges; and since [1, p. 262, formula (3.4)]

$$\left|\frac{X_p(z)}{Y_p(z)}\right| \leqq \frac{1}{y} \qquad \text{for } y = I(z) > 0, \tag{3.12}$$

it follows that the second of the series (2.2) diverges for $I(z)>0$. Therefore [1, p. 263, formula (3.12)], the radius $r_p(z)$ of the circle $K_p(z)$ tends to 0 as p tends to ∞. This implies that there is only one function $f_0(z)$ which for $I(z)>0$ has its values in all the circles $K_p(z)$. The reciprocal (ρ_{pq}) of $J+zI$ given by (2.3) with $f(z)=f_0(z)$ is bounded for $I(z)>0$. It is required to show that any other reciprocal is unbounded for at least one z in $I(z)>0$.

We consider an arbitrary reciprocal of $J+zI$ in $I(z)>0$. This must be given by (2.3). If $f(z)\not\equiv f_0(z)$ for $I(z)>0$, there must exist a value $z=z_0$, $I(z_0)>0$, such that

$$\left| f(z_0) - \frac{X_p(z_0)}{Y_p(z_0)}\right| \geqq k,$$

for all sufficiently large values of p, k being a positive constant. This follows from the fact that $X_p(z_0)/Y_p(z_0)$ is in the circle $K_{p-1}(z_0)$. Hence by (2.3), $|\rho_{pq}(z_0)|^2 \geqq |Y_q(z_0)|^2 k^2 \cdot |Y_p(z_0)|^2$, for each q and for all sufficiently large values of p. Since $|Y_q(z_0)|>0$ by (3.12), and since the series $\sum |Y_p(z_0)|^2$ is divergent, it follows that the series

$$\sum_{p=1}^{\infty} |\rho_{pq}(z_0)|^2$$

is divergent. Therefore the matrix $(\rho_{pq}(z_0))$ is unbounded.

Bibliography

1. J. J. Dennis and H. S. Wall, *The limit-circle case for a positive definite J-fraction*, Duke Math. J. vol. 12 (1945) pp. 255–273.

2. E. Hellinger, *Zur Stieltjesschen Kettenbruchtheorie*, Math. Ann. vol. 86 (1922) pp. 18–29.

3. E. D. Hellinger and H. S. Wall, *Contributions to the analytic theory of continued fractions and infinite matrices*, Ann. of Math. (2) vol. 44 (1943) pp. 103–127.

4. H. S. Wall and Marion Wetzel, *Contributions to the analytic theory of J-fractions*, Trans. Amer. Math. Soc. vol. 55 (1944) pp. 373–392.

5. ———, *Quadratic forms and convergence regions for continued fractions*, Duke Math. J. vol. 11 (1944) pp. 89–102.

Illinois Institute of Technology