THE ASANO POSTULATES FOR THE INTEGRAL DOMAINS OF A LINEAR ALGEBRA

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1. **Introduction.** The multiplicative ideal theory for a noncommutative ring A as developed by Asano[1] postulates the existence in A of a maximal bounded order R which satisfies the maximal chain condition for two-sided R-ideals contained in R and the minimal chain condition for one-sided R-ideals in R containing any fixed two-sided R-ideal. Let A be a separable algebra over the field P, and let P be the quotient field of the domain of integrity g. It has been shown [2, pp. 123–126] that if g has a Noether ideal theory, then a maximal domain of g-integers exists in A and satisfies the conditions of the Asano theory. It is the purpose of this paper to prove that the condition of separability can be removed from A and that it need only be postulated that A shall have an identity.

2. **Subgroups of direct sums.** Let G be a commutative group with operator domain Ω. Let G be the direct sum of the Ω-subgroups $G_1, G_2, \cdots, G_n$. We shall write $G = G_1 + G_2 + \cdots + G_n$. The direct summand G_i gives rise to a projection α_i which is an endomorphism of G on G_i: if $g = g_1 + g_2 + \cdots + g_n$, $g_j \in G_j$, then $\alpha_i g = g_i$. The sum $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ is the identity operator I. Furthermore the sum of any subset of the projections $\alpha_1, \alpha_2, \cdots, \alpha_n$ is a projection. We shall label in particular the operators $\delta_i = \sum_{j=1}^{i} \alpha_j$. Then $\delta_1 = \alpha_1$, and $\delta_n = I$. In general $\delta_{i+1} = \delta_i + \alpha_{i+1}$. If $\omega \in \Omega$, then $\omega\alpha_i = \alpha_i\omega$, and as a result $\omega\delta_i = \delta_i\omega$; that is, α_i and δ_i are Ω-operators. It follows that $\alpha_i H$ and $\delta_i H$ are Ω-subgroups whenever H is an Ω-subgroup.

LEMMA 1. *Let the commutative group $G = G_1 + G_2 + \cdots + G_n$ contain the Ω-subgroups H and K. If $H \supseteq K$, then $\alpha_i H \supseteq \alpha_i K$, $\delta_i H \supseteq \delta_i K$, and $\delta_i H \cap G_i \supseteq \delta_i K \cap G_i$.*

Since $H \supseteq K$, the image $\alpha_i K$ of K under the homomorphism of H on $\alpha_i H$ must be contained in $\alpha_i H$. By the same argument $\delta_i H \supseteq \delta_i K$, and therefore $\delta_i H \cap G_i \supseteq \delta_i K \cap G_i$.

LEMMA 2. *Let the commutative group $G = G_1 + G_2 + \cdots + G_n$ contain the Ω-subgroups H and K. If $H \supseteq K$ and if $\alpha_i H = \alpha_i K$, $\delta_i H \cap G_i = \delta_i K \cap G_i$, then $H = K$.*

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[1] Cf. Asano [1], Jacobson [2]. We use here the formulation of these postulates given by Jacobson. Numbers in brackets refer to the references at the end of the paper.

Since $\delta_1 = \alpha_1$ and $\alpha_1 H = \alpha_1 K$, it follows that $\delta_1 H = \delta_1 K$.

We shall assume that $\delta_i H = \delta_i K$ and prove that under this assumption $\delta_{i+1} H = \delta_{i+1} K$. Since $\delta_{i+1} K = (\delta_i + \alpha_{i+1}) K \subseteq \delta_i K + \alpha_{i+1} K$ and $\delta_{i+1} K \subseteq \delta_{i+1} H$, it is obvious that $\delta_{i+1} K \subseteq (\delta_i K + \alpha_{i+1} K) \cap \delta_{i+1} H$. On the other hand let $\delta_i k_1 + \alpha_{i+1} k_2$ be an element of $\delta_i K + \alpha_{i+1} K$ contained in $\delta_{i+1} H$. Consider that $(\delta_i + \alpha_{i+1}) k_1$ is an element of $\delta_{i+1} K$ and therefore an element of $\delta_{i+1} H$. Then $\delta_{i+1} H$ contains $\delta_i k_1 + \alpha_{i+1} k_2 - (\delta_i + \alpha_{i+1}) k_1 = \alpha_{i+1}(k_2 - k_1)$ which lies in $\delta_{i+1} H \cap G_{i+1} = \delta_{i+1} K \cap G_{i+1} \subseteq \delta_{i+1} K$. It follows immediately that $\delta_i k_1 + \alpha_{i+1} k_2 = (\delta_i + \alpha_{i+1}) k_1 + \alpha_{i+1}(k_2 - k_1)$ lies in $\delta_{i+1} K$, and $\delta_{i+1} H \cap (\delta_i K + \alpha_{i+1} K) = \delta_{i+1} K$. However, since $\delta_i K = \delta_i H$ and $\alpha_{i+1} K = \alpha_{i+1} H$, then $\delta_i K + \alpha_{i+1} K = \delta_i H + \alpha_{i+1} H$, and $\delta_{i+1} K = \delta_{i+1} H \cap (\delta_i H + \alpha_{i+1} H) = \delta_{i+1} H$.

The lemma follows by finite induction; for $\delta_n H = H$, $\delta_n K = K$.

LEMMA 3. *Let the commutative group $G = G_1 + G_2 + \cdots + G_n$ contain the Ω-subgroup H. Let γ be an automorphism of G contained in the centrum of Ω. Then $H \supseteq \gamma H$, $\alpha_i H \supseteq \alpha_i(\gamma H) = \gamma(\alpha_i H)$, and $\delta_i H \cap G_i \supseteq \delta_i(\gamma H) \cap G_i = \gamma(\delta_i H \cap G_i)$.*

The automorphism γ lies in the centrum of Ω and therefore γH will be an Ω-subgroup of H. It follows by Lemma 1 that $\alpha_i H \supseteq \alpha_i(\gamma H)$, $\delta_i H \supseteq \delta_i(\gamma H)$, and $\delta_i H \cap G_i \supseteq \delta_i(\gamma H) \cap G_i$. Since γ lies in Ω and α_i and δ_i are Ω-operators, $\alpha_i(\gamma H) = \gamma(\alpha_i H)$ and $\delta_i(\gamma H) = \gamma(\delta_i H)$.

It remains to prove that $\delta_i(\gamma H) \cap G_i = \gamma(\delta_i H \cap G_i)$. Consider that $\gamma G_i = \gamma \alpha_i G = \alpha_i \gamma G = \alpha_i G = G_i$. Then $\delta_i(\gamma H) \cap G_i = \delta_i(\gamma H) \cap \gamma G_i = \gamma(\delta_i H) \cap \gamma G_i$. Let $\gamma \delta_i h = \gamma g_i$; γ is an automorphism, and $\delta_i h = g_i$. It follows that $\gamma(\delta_i H) \cap \gamma G_i \supseteq \gamma(\delta_i H \cap G_i)$. But certainly $\gamma(\delta_i H \cap G_i) \subseteq \gamma(\delta_i H) \cap \gamma G_i$ for any operator γ.

THEOREM 1. *Let G be a commutative Ω-group, and let Ω contain an automorphism γ in its centrum. Let G be the direct sum of the Ω-subgroups $G_1, G_2, \cdots, G_n$, and let G contain the Ω-subgroup H. If for every Ω-subgroup A_i of G_i the Ω-group $A_i/\gamma A_i$ satisfies the minimal (maximal) chain condition for Ω-subgroups of $A_i/\gamma A_i$, then the Ω-group $H/\gamma H$ satisfies the minimal (maximal) chain condition for Ω-subgroups of $H/\gamma H$.*

A chain of Ω-subgroups

$$\text{(A)} \qquad H \supset H_1 \supset H_2 \supset \cdots \supset \gamma H$$

implies, by Lemmas 1 and 3, the existence of the $2n$ chains

$$\alpha_i H \supseteq \alpha_i H_1 \supseteq \alpha_i H_2 \supseteq \cdots \supseteq \gamma(\alpha_i H), \qquad i = 1, 2, \cdots, n,$$

$$\text{(B)} \quad \delta_i H \cap G_i \supseteq \delta_i H_1 \cap G_i \supseteq \delta_i H_2 \cap G_i \supseteq \cdots \supseteq \gamma(\delta_i H \cap G_i), \qquad i = 1, 2, \cdots, n.$$

Lemma 2 implies that if the chain (A) is infinite, at least one of the chains (B) must be nontrivially infinite. If the minimal chain condition fails in $H/\gamma H$, it must fail in one of the groups $\alpha_i H/\gamma(\alpha_i H)$ or $\delta_i H \cap G_i/\gamma(\delta_i H \cap G_i)$ where $\alpha_i H$ and $\delta_i H \cap G_i$ are Ω-subgroups of G_i.

The statement of the theorem for maximal chains follows by the same argument.

3. **Chains of g-modules.** Let g be a domain of integrity with Noether ideal theory. This implies that in g every ideal is the product of powers of prime ideals and that a prime ideal is divisorless. If P is the quotient field of g, fractional ideals are defined in P. The set of all ideals in P forms a group under multiplication. In particular if $\mathfrak{a}$ is an ideal, $\mathfrak{a}^{-1}$ will exist such that $\mathfrak{a}\mathfrak{a}^{-1}=g$, and if $\mathfrak{a}\mathfrak{c}=\mathfrak{b}\mathfrak{c}$, then $\mathfrak{a}=\mathfrak{b}$.

A g-module in P is a set of elements of P which forms a group under addition and is closed under multiplication by elements of g. The g-module $\mathfrak{a}$ is an ideal if $\alpha\mathfrak{a}\subseteq g$ for some element $\alpha\neq 0$ of g. The product of an ideal contained in g and a g-module $\mathfrak{a}$ is contained in $\mathfrak{a}$. If $\mathfrak{a}\supset\mathfrak{b}$, the group $\mathfrak{a}/\mathfrak{b}$ is a g-module (not contained in P).

LEMMA 4. *If g has a Noether ideal theory, and if $\mathfrak{a}$ is a g-module in the quotient field P of g, the g-module $\mathfrak{a}/\alpha\mathfrak{a}$ has a composition series for any element $\alpha\neq 0$ of g.*

Let $\mathfrak{a}$ be a g-module contained in P, and let α be an element not equal to 0 of g. If the principal ideal (α) has the factorization $\mathfrak{p}_1^{r_1}\mathfrak{p}_2^{r_2}\cdots\mathfrak{p}_s^{r_s}$ in g, we shall prove that the chain of g-modules

$$\mathfrak{a}\supseteq\mathfrak{p}_1\mathfrak{a}\supseteq\mathfrak{p}_1^2\mathfrak{a}\supseteq\cdots\supseteq\mathfrak{p}_1^{r_1}\mathfrak{a}\supseteq\mathfrak{p}_1^{r_1}\mathfrak{p}_2\mathfrak{a}\supseteq\cdots\supseteq\alpha\mathfrak{p}_s^{-1}\mathfrak{a}\supseteq\alpha\mathfrak{a}$$

allows no nontrivial refinement. The series

$$\mathfrak{a}/\alpha\mathfrak{a}\supseteq\mathfrak{p}_1\mathfrak{a}/\alpha\mathfrak{a}\supseteq\mathfrak{p}_1^2\mathfrak{a}/\alpha\mathfrak{a}\supseteq\cdots\supseteq\mathfrak{p}_1^{r_1}\mathfrak{a}/\alpha\mathfrak{a}$$
$$\supseteq\mathfrak{p}_1^{r_1}\mathfrak{p}_2\mathfrak{a}/\alpha\mathfrak{a}\supseteq\cdots\supseteq\alpha\mathfrak{p}_s^{-1}\mathfrak{a}/\alpha\mathfrak{a}\supseteq(0)$$

will include a composition series for $\mathfrak{a}/\alpha\mathfrak{a}$.

Let $\mathfrak{p}$ be a prime ideal in g, and let $\mathfrak{b}$ be a g-module contained in P. Assume that between $\mathfrak{b}$ and $\mathfrak{p}\mathfrak{b}$ there lies a g-module $\mathfrak{c}$ equal to neither: $\mathfrak{b}\supset\mathfrak{c}\supset\mathfrak{p}\mathfrak{b}$. Then there is an element β of $\mathfrak{b}$ not contained in $\mathfrak{c}$ and an element γ of $\mathfrak{c}$ not contained in $\mathfrak{p}\mathfrak{b}$. We form the chain of ideals of P: $(\beta,\gamma)\supset(\mathfrak{p}\beta,\gamma)\supset\mathfrak{p}(\beta,\gamma)$. Since $\mathfrak{p}\beta\subseteq\mathfrak{p}\mathfrak{b}\subset\mathfrak{c}$ and $\gamma\in\mathfrak{c}$, $(\mathfrak{p}\beta,\gamma)\subseteq\mathfrak{c}$. But β is not an element of $\mathfrak{c}$, and therefore $(\mathfrak{p}\beta,\gamma)$ and (β,γ) are distinct. Since $\mathfrak{p}(\beta,\gamma)\subseteq\mathfrak{p}\mathfrak{b}$ and γ is not an element of $\mathfrak{p}\mathfrak{b}$, $\mathfrak{p}(\beta,\gamma)$ and $(\mathfrak{p}\beta,\gamma)$ are distinct. It would follow that $g\supset(\mathfrak{p}\beta,\gamma)(\beta,\gamma)^{-1}\supset\mathfrak{p}$ is a chain of distinct ideals in g. However, the prime ideal $\mathfrak{p}$ is divisorless. It follows that $\mathfrak{a}\supset\mathfrak{p}\mathfrak{a}$ allows no nontrivial refinement.

If M is a P-module with linearly independent P-basis $x_1, x_2, \cdots, x_n$ we shall write $M=Px_1+Px_2+\cdots+Px_n$.

THEOREM 2. *Let $M=Px_1+Px_2+\cdots+Px_n$ contain the g-module N. Then if γ is an element not equal to* 0 *of g, the g-module $N/\gamma N$ has a composition series.*

The module N is a g-submodule of the direct sum $Px_1+Px_2+\cdots+Px_n$. The element γ of g is an automorphism of M, and the operator domain g is commutative. Lemma 4 assures us that for every g-subgroup $\mathfrak{a}x_i$ of Px_i the g-module $\mathfrak{a}x_i/\gamma(\mathfrak{a}x_i)\cong\mathfrak{a}/\gamma\mathfrak{a}$ has a composition series. The conditions of Theorem 1 are satisfied, and the g-module $N/\gamma N$ must have a composition series.

4. Orders of finite linear algebras. We shall again assume that g is a domain of integrity with Noether ideal theory and that P is the quotient field of g. We consider a linear algebra A with identity e of order n over the field P.

An order R of A which contains g can be defined to be a subring of A which contains g and a basis for A [2, p. 124]. We shall consider only orders of A which contain g. A left (right) R-ideal of R is a submodule $\mathfrak{M}$ of R such that $R\mathfrak{M}\subseteq\mathfrak{M}$ ($\mathfrak{M}R\subseteq\mathfrak{M}$) and which contains a regular element of A. Then $\mathfrak{M}$ contains an element $\gamma\neq 0$ of g and contains the two-sided ideal γR: every order R is bounded. Since R contains g, R and every R-ideal of R are g-modules.

THEOREM 3. *Let g be a domain of integrity with Noether ideal theory, and let P be the quotient field of g. If A is a linear algebra with identity of finite order over P, every order of A which contains g will satisfy the maximal condition for any chain of left* (*right*) *R-ideals contained in R and the minimal condition for any chain of left* (*right*) *R-ideals in R containing a fixed left* (*right*) *R-ideal.*

We may consider the algebra A to be the P-module $Px_1+Px_2+\cdots+Px_n$ where $x_1, x_2, \cdots, x_n$ constitute a linearly independent basis for A over P, and R as a g-submodule of A. An R-ideal $\mathfrak{M}$ of R contains an element $\gamma\neq 0$ of g so that $R\supseteq\mathfrak{M}\supseteq\gamma R$. By Theorem 2 every chain of g-modules between R and γR must be finite. In particular a chain of R-ideals between R and $\mathfrak{M}$ must be finite since an R-ideal is a g-module if R contains g.

Two orders R and R' are said to be equivalent if there exist regular elements a, b, c, d of A such that $aRb\subseteq R'$, $cR'd\subseteq R$. An order is said to be maximal if it is contained in no equivalent order.

The Asano treatment of the ideal theory of a class of equivalent orders depends on three postulates:

I. There exists a maximal bounded order R in the class.

II. The minimal chain condition holds for left R-ideals in R which contain a fixed two-sided R-ideal.

III. The maximal chain condition holds for two-sided R-ideals contained in R.

In Theorem 3 we have shown that postulates II and III are satisfied by any order of A which contains g. If a maximal order exists, it must be bounded since every order is bounded.

An order of A which contains g and contains only integral elements of A is called an integral domain. A maximal integral domain is an integral domain which is contained in no other integral domain.

LEMMA 5. *If the order R contains g and is equivalent to the integral domain S, then R is an integral domain.*

Since R is equivalent to S there exist regular elements a, b such that $aRb\subseteq S$. Since R is an order of A there exists in g an element $\beta\neq 0$ such that βb^{-1} is an element of R. Then $\beta b^{-1}R\subseteq R$. Similarly S, which is an order of A, must contain αa^{-1} for some element $\alpha\neq 0$ of g, and $\alpha S a^{-1}\subseteq S$. Then

$$\alpha[a(\beta b^{-1}R)b]a^{-1} \subseteq \alpha[aRb]a^{-1} \subseteq \alpha S a^{-1} \subseteq S,$$

or

$$(ab^{-1})(\alpha\beta)R(ba^{-1}) \subseteq S.$$

Set $\alpha\beta=\gamma$, $ab^{-1}=c$; then $c(\gamma R)c^{-1}\subseteq S$, and $\gamma R\subseteq c^{-1}Sc$ where c is a regular element of A. It follows that γR consists only of integral elements of A.

Let r be an element of R. Let $g[r]$ indicate the polynomial domain generated by r with coefficients in g; $g[r]$ is a commutative ring contained in R. Further $\gamma g[r]$ is a ring of integers. If we consider that $g[r]$ is a g-module contained in the P-module $A=Px_1+Px_2+\cdots+Px_n$ we may apply Theorem 2 to $g[r]$ and obtain that every chain of g-modules between $\gamma g[r]$ and $g[r]$ is finite. If H is the union of g and $\gamma g[r]$, H is a ring of integers, and $\gamma g[r]\subseteq H\subseteq g[r]$. Since $g\subset H$, the chain of H-modules

$$H \subseteq Hr \subseteq (Hr, Hr^2) \subseteq \cdots \subseteq g[r]$$

is a chain of g-modules between H and $g[r]$ and must be finite in length. It follows that r satisfies an equation $r^k=h_1r^{k-1}+h_2r^{k-2}+\cdots+h_kr$ with coefficients in H. Then r is g-integral, and R is an integral domain [3, p. 90].

COROLLARY. *A maximal integral domain S is a maximal order in the class of orders equivalent to S.*

We can now establish the existence in A of a maximal order by the following argument: Let all integral domains S_α of A be well-ordered. Construct a chain

$$S \subset S_{\sigma_1} \subset S_{\sigma_2} \subset \cdots$$

of domains containing a fixed domain S by choosing S_{σ_1} to be the first which contains S, S_{σ_2} to be the first which contains S_{σ_1}, and so on. The union R of the S_{σ_i} will be a maximal integral domain and, by the above corollary, R is a maximal order. The class of orders equivalent to R will satisfy the Asano postulates.

THEOREM 4. *Let g be a domain of integrity with Noether ideal theory, and let P be the quotient field of g. Every linear algebra with identity of finite order P contains a nontrivial class of orders which satisfy the Asano postulates and which contain only integral elements of the algebra.*

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