TOEPLITZ METHODS WHICH SUM A GIVEN SEQUENCE

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The following note arose out of discussions of a paper by Agnew, but is, however, self-contained.

THEOREM. *Let $\{x_n\}$ be a bounded divergent sequence. Suppose that $\{y_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$. Then $\{y_n\}$ is of the form $\{cx_n+a_n\}$ where $\{a_n\}$ is convergent.*

PROOF. For typographical convenience we shall often write $x(n)$ for x_n, and so on. Let $\{x(n_k)\}$, $k=1, 2, \cdots$, be any convergent subsequence of $\{x_n\}$. Then $\{x_n\}$ is summable by the matrix $(a(n, k))$ where $a(n, k)=1$ for $n=n_k$ and $a(n, k)=0$ for $n\neq n_k$. Hence $\{y(n_k)\}$ is also convergent.

Let $\{n_k'\}$ and $\{n_k''\}$ be sequences of integers such that $n_k'\neq n_k''$ for all k and

$$\lim_{k\to\infty} x(n_k') = A, \qquad \lim_{k\to\infty} x(n_k'') = B, \qquad A\neq B.$$

These sequences $\{n_k'\}$ and $\{n_k''\}$ will be held fixed throughout the rest of the argument. Then the sequences $\{y(n_k')\}$ and $\{y(n_k'')\}$ are also convergent, say to α and β respectively. Let $\{x(n_k)\}$ be an arbitrary convergent subsequence of $\{x_n\}$ with the limit C. Let λ and μ be determined by the equations

$$\lambda+\mu = 1, \qquad \lambda A+\mu B = C.$$

Then the matrix $(b(n, k))$ with

$$b(n, k) = \begin{cases} \lambda, & n = n_k', \quad k \text{ even}, \\ \mu, & n = n_k'', \quad k \text{ even}, \\ 1, & n = n_k, \quad k \text{ odd}, \\ 0, & \text{for all other values of } n \text{ and } k, \end{cases}$$

sums $\{x_n\}$ to the limit C. Hence it also sums $\{y_n\}$, that is

$$\lim_{k\to\infty} y(n_k) = \lim_{k\to\infty} (\lambda y(n_k') + \mu y(n_k'')) = \lambda\alpha + \mu\beta.$$

Note that the numbers λ and μ depend only on C and not on the particular subsequence $\{x(n_k)\}$ converging to C, and hence $\lim_{k\to\infty} y(n_k)$

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also depends only on C, and is, in fact, a linear function of C.

We now determine constants m and a from the equations

$$\alpha = mA + a, \qquad \beta = mB + a.$$

Let $\{n_k\}$ be an arbitrary sequence of positive integers, and let $\{n_k'''\}$ be a subsequence of $\{n_k\}$ such that $\{x(n_k''')\}$ converges, say to C. We determine λ and μ as before Then

$$\begin{aligned}\lim_{k\to\infty} (y(n_k''') - mx(n_k''')) &= \lambda\alpha + \mu\beta - mC \\ &= \lambda(mA + a) + \mu(mB + a) - mC = a.\end{aligned}$$

Thus every subsequence of $\{y_n - mx_n\}$ contains a subsubsequence converging to a. Hence $\lim_{n\to\infty} (y_n - mx_n) = a$, which proves our theorem.

COROLLARY. *If $\{x_n\}$ and $\{y_n\}$ are bounded divergent sequences, and $\{y_n\}$ is summable by every regular Toeplitz method which sums $\{x_n\}$, then $\{x_n\}$ is summable by every regular Toeplitz method which sums $\{y_n\}$.*

By a theorem of Agnew,[1] there is no single Toeplitz method which has the sequences of the form $\{cx_n + a_n\}$ as its convergence field. The above theorem shows, however, that this set of sequences is the common part of the convergence fields of Toeplitz methods which sum $\{x_n\}$.

Added November 11, 1945 We have just had the opportunity of seeing the paper of A. Brudno, *Summation of bounded sequences by matrices*, Rec. Math. (Mat. Sbornik) N.S. vol. 16 (1945), pp. 191–247. From the English summary it seems that our result is contained in his Theorem 11, p. 236. His Theorem 11 can clearly be proved by our method. It is difficult to compare the simplicity of the proofs.

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[1] R. P. Agnew, *Convergence fields of methods of summability*, Ann. of Math. vol. 46 (1945) pp. 93–101.