A NOTE ON AXIOMATIC CHARACTERIZATION OF FIELDS

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Since publication of our paper, *Axiomatic characterization of fields by the product formula for valuations*,[1] we have found that the fields of class field theory can be characterized by somewhat weaker axioms; we can drop the assumption, in Axiom 1, that $|\alpha|_{\mathfrak{p}}=1$ for all but a finite number of $\mathfrak{p}$, replacing it by the assumption that the product of all valuations converges absolutely to the limit 1 for all α.

Our original proof can be adapted to the new axiom with a few modifications, which we shall describe here. In §2, we keep Axiom 1 for reference and introduce:

AXIOM 1*. *There is a set $\mathfrak{M}$ of prime divisors $\mathfrak{p}$ and a fixed set of valuations $|\ |_{\mathfrak{p}}$, one for each $\mathfrak{p}\in\mathfrak{M}$, such that, for every $\alpha\neq 0$ of k, the product $\prod_{\mathfrak{p}}|\alpha|_{\mathfrak{p}}$ converges absolutely to the limit* 1. (*That is, the series $\sum_{\mathfrak{p}}\log|\alpha|_{\mathfrak{p}}$ converges absolutely to* 0.)

We must then omit the statement that there are only a finite number of archimedean primes, since this does not follow immediately from 1*; instead of it, we use the fact that $\sum_{\mathfrak{p}_\infty}\rho(\mathfrak{p}_\infty)$ and $\sum_{\mathfrak{p}_\infty}\lambda(\mathfrak{p}_\infty)$ converge absolutely. These quantities are defined on p. 480; the convergence follows from the fact that the product over all $\mathfrak{p}_\infty$ of $|1+1|_{\mathfrak{p}_\infty}$ must converge absolutely. Also, we must temporarily broaden the definition of "parallelotope" so as to permit a parallelotope to be defined by any valuation vector $\mathfrak{a}$ for which $\prod_{\mathfrak{p}}|\mathfrak{a}|_{\mathfrak{p}}$ converges absolutely (rather than restricting $\mathfrak{a}$ to be an idèle). In the statement of Axiom 2 we must replace "Axiom 1" by "Axiom 1*," Theorem 2, however, is left unchanged, together with Lemmas 4, 5, and 6, which are needed only to prove it; this theorem shows that the fields of class field theory really satisfy Axiom 1, so that at the end of the whole proof we shall find that Axiom 1 is a consequence of Axioms 1* and 2.

In §3, k is assumed to be any field for which Axioms 1* and 2 hold. Lemma 8 holds under assumption of Axiom 1*, for our slightly more general parallelotopes; in its proof we have only to note, in case of archimedean primes, that the product $\prod_{\mathfrak{p}_\infty}4^{\rho(\mathfrak{p}_\infty)}$ converges absolutely. In Lemma 9, property 2 must be replaced by:

2*. $|\alpha|_{\mathfrak{p}_\infty}\leqq B_{\mathfrak{p}_\infty}|y|_{\mathfrak{p}_\infty}$, *with a set of constants $B_{\mathfrak{p}_\infty}$ for which $\prod_{\mathfrak{p}_\infty}B_{\mathfrak{p}_\infty}$ converges absolutely.*

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To prove existence of these constants, let, at each $\mathfrak{p}_\infty$, $M_{\mathfrak{p}_\infty}$ be the maximum of $|\alpha_i|_{\mathfrak{p}_\infty}$ for $i=1 \cdots l$; then $\prod_{\mathfrak{p}_\infty} M_{\mathfrak{p}_\infty}$ converges to a finite limit. Take $B_{\mathfrak{p}_\infty}=M_{\mathfrak{p}_\infty} l^{\lambda(\mathfrak{p}_\infty)}$; since $\sum_{\mathfrak{p}_\infty}\lambda(\mathfrak{p}_\infty)$ was absolutely convergent, our conclusion follows.

Lemma 10 holds as stated, although the set of $\mathfrak{p}_\infty$ is not now known to be finite. But as soon as we have proved that n is finite, it follows from Theorem 2 that our original Axiom 1 holds, so no further changes are necessary. (The theorems about parallelotopes in §4 hold only for parallelotopes defined by ideal elements.)

It is easy to construct an example of a field which satisfies Axiom 1* but does not satisfy Axiom 1 (nor, of course, Axiom 2). Let $k=R(x, z)$ be the set of all rational functions of x and z over the rational field. Let $k_0=R(x)$, consider k as the set $k_0(z)$ of all rational functions of z with k_0 as constant field, and denote by $\mathfrak{M}_0$ the set of all divisors which are trivial on k_0. We construct $\mathfrak{M}_0$, and define the set of normed valuations, exactly as in the proof of Lemma 6 of our original paper (pp. 477–479). Let $V_0(A)=\prod\|A\|_{p_0}$ where the product is taken over all $p_0 \in \mathfrak{M}_0$; by Lemma 6, $V_0(A)=1$ for all $A\in k$.

Now let $x_1=x+z$, $x_2=x+2z, \cdots, x_i=x+iz, \cdots$; let $k_i=R(x_i)$ and for each i construct the sets $\mathfrak{M}_i$ of divisors p_i by repeating exactly the above process with k_0 replaced by k_i. The products $V_i(A)$ are all equal to 1. These sets $\mathfrak{M}_i$ are by no means disjoint; for example one can easily see that the irreducible polynomial z defines the same valuation in each $\mathfrak{M}_i$. However, it is unnecessary to explore these duplications in detail; we shall need only the facts that the valuations $p_{i\infty}$ and $p_{j\infty}$ are inequivalent for $i\neq j$, and are not equivalent to any of the finite p_ν. Namely, $x_i=x+iz=x_j+(i-j)z$ has value 1 at $p_{i\infty}$, but value $q>1$ at all $p_{j\infty}$ with $j\neq i$. And z has value $q>1$ at all $p_{i\infty}$, but has value $\leqq 1$ at all finite p_ν.

To construct our example, let ϵ_ν $(\nu=0, 1, 2, \cdots)$ be an infinite sequence of positive numbers whose sum is finite. Form the product

$$\prod \|A\|_{p_i}^{\epsilon_i}$$

over all $p_i\in\mathfrak{M}_i$, all i, and in this product unite each set of equivalent valuations into a single valuation. The exponents insure the convergence of the infinite products involved in this step. To show that the whole product is absolutely convergent for each $A\in k$, write A in the form $A=g(x, z)/h(x, z)$ where g and h are polynomials with rational coefficients. If N and M are the maximum degrees in x and z, respectively, for both numerator and denominator, then A can be written in the form $g_i(z)/h_i(z)$, where numerator and denominator are polynomials in z with coefficients in k_i, and are of degree at most $N+M$ in z. It follows from this that, for fixed A, the number of factors of $V_i(A)$ which are greater than 1 (or which are less than 1) is bounded, and their size is bounded; and this bound is uniform for all i. Hence the exponents ϵ_i insure absolute convergence. Finally, we note that our product, applied to z, contains an infinity of factors different from 1.

Taking the product over sets $\mathfrak{M}_0$ and $\mathfrak{M}_1$ only gives an example in which Axiom 1 is satisfied but Axiom 2 is not; for the field of constants with respect to $\mathfrak{M}_0 \cup \mathfrak{M}_1$ is the rational field $k_0 \cap k_1$.

To get an example of a field possessing a valuation satisfying Axiom 2, but such that this valuation cannot be contained in any set $\mathfrak{M}$ satisfying Axiom 1, take the p-adic closure of either the rational field or any of the fields $k_0(z)$ of our original paper, with p any of the divisors of Lemma 6. Because of Theorem 3, such an $\mathfrak{M}$ cannot exist.

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