BOOK REVIEWS

The theory of rings. By Nathan Jacobson. (Mathematical Surveys, no. 2.) New York, American Mathematical Society, 1943. 150 pp. $2.25.

In recent years the theory of rings has been one of the centers of most vigorous mathematical life. Although originally an outgrowth of the theory of algebras, it has made itself completely independent of its origin. This was necessitated by two considerations. Firstly, it appeared that the special hypotheses inherent to the theory of algebras were not needed for the greater part of the theory of rings. The latter theory, thus stripped of unnecessary encumbrances, became more general and at the same time simpler, clearer and more elegant. Secondly, there are some important applications of the theory of rings that do not fit into the framework of the theory of algebras, such as the applications to the rings of endomorphisms of abelian operator groups.

In the book under consideration a comprehensive account is given of the theory of rings. In view of the fact that mathematicians from all over the world have contributed toward the growth of the theory and that their results are scattered over all the international mathematical periodicals, the collection of the material is by itself no mean task (Jacobson's bibliography covers eight pages of fine print). But it is even more important and difficult to obtain a unified treatment of such a host of different methods and points of view. In this Jacobson has succeeded admirably by means of the methodological principle which he uses. The knowledge obtained in the abstract theory of rings is first applied to the study of abelian operator groups over rings (Emmy Noether's representation moduli). The structural theory of abelian operator groups is next used for an investigation of their rings of endomorphisms. The cycle is closed by means of the following theorem which permits the application of the theory of endomorphism rings to the abstract theory of rings: If R is an abstract ring with an identity element, we denote by a_r (by a_l) for a in R the linear transformation which maps the element x in R onto xa (onto ax); if we consider the additive group R_+ of R as an operator group with respect to the a_l, the ring of the a_r is just the full endormorphism ring and it is essentially the same as the original abstract ring R.

It would lead us too far to give a halfway complete description of the wealth of material covered in this book. It must suffice to mention some of the more salient facts.

In the first chapter the generalities of the theory of operator groups are developed, including a lucid treatment of the uniqueness theorem for direct decompositions which is based on Fitting's Lemma. These two results are used continuously in the sequel.

The general structure theory of rings has been developed solely on the basis of the descending chain condition for ideals. This is made possible by proving C. Hopkins' two theorems which assert the existence of the radical (and its nilpotency) in such rings and the validity of the ascending chain condition for ideals, provided the ring contains an identity. Principal ideal domains enjoy many surprising properties. Most significant are the elementary divisor theorem of Jacobson-Teichmüller and the corresponding invariance theorem of Nakayama.

The structure of finitely generated abelian groups over principal ideal rings is elucidated by proving that such groups are direct sums of indecomposable cyclic groups, and that their structure is completely determined by the bounds of the orders of these cyclic direct summands which are thus shown to be characteristic invariants.

The whole book culminates in the chapter on multiplicative ideal theory. Almost every result is applied here and many concepts find an illustration in this theory. It is shown that for the two-sided ideals in an order one may obtain unique factorization in the classical sense, including the commutativity of multiplication, and that under comparatively simple necessary conditions on the orders in the ring one may construct Brandt's groupoid of ideals. This arithmetic theory of ideals may be considered, in a way, as the conclusion of a development that started with Emmy Noether's famous five axioms and with Dickson's *Algebras and their arithmetics.*

Numerous applications are given in the book: the ring which is obtained by adjoining a semi-linear transformation to a field, the proof of the complete reducibility of a finite group of semi-linear transformations provided the characteristic of the field is not a divisor of the subgroup of ordinary linear transformations, R. Brauer's group of algebras over a field, the author's Galois theory of non-commutative fields.

This very excellent treatment of the theory of rings is more than a compendium. For not only does it offer a résumé of the contents of this theory, but it gives at the same time a very instructive introduction into the working methods used here which will be helpful to others than the algebraists. Since the book is, apart from a very few exceptions, quite self-contained, not much previous knowledge is needed, although it seems desirable to be familiar with the ways of thinking practised in abstract algebra. Thus the book should not only be indispensable to every worker in the theory of rings, but may also be used in connection with an introductory course in abstract algebra.

REINHOLD BAER

An essay on the psychology of invention in the mathematical field. By Jacques Hadamard. Princeton University Press, 1945. 143 pp. $2.00.

Professor Hadamard points out at the beginning of his little book that he is handicapped in the study it deals with by not being a psychologist. Perhaps I should point out that I am handicapped in reviewing him by being neither a psychologist nor a mathematician. But as he bravely goes on, so must I; both of us converging on that question of extraordinary interest in the history of ideas: How do great discoveries and inventions come about?

Hadmard's answer—limited, of course, to the mathematical field—is based on a variety of evidence: the testimony of contemporary mathematicians, the writings of previous psychologists, philosophers and scientists, the interpretation of certain characteristics (logical or intuitive) in the work of famous discoverers and, finally, the author's own minute introspection.

From a careful analysis and comparison of these diverse materials, Professor Hadamard concludes that the general pattern of invention, or, as it might also be put, of original work, is three-fold: conscious study, followed by unconscious maturing, which leads in turn to the moment of insight or illumination. Thereupon another period of conscious work ensues, the purpose of which is to achieve a synthesis of several elements: the novel idea, its logically deduced consequences including proof, and the traditional knowledge to which the new item is added.

Hadamard's investigation, modest and tentative as are its results, seems to me of capital importance in the realm of criticism and cultural history. For what he has done is to show that the human mind tends to behave much the same way whenever it invents, whether in mathematical or in poetic form—a conclusion which does not deny differences of temperament. Our author, on the contrary, is at pains to distinguish among types of mathematical geniuses. He classes them as logical or intuitive, concrete or abstract, yet with enough flexibility to allow for deceptive appearances and for the overlapping of categories. But it is clear in the end that in any process of creation there lurks a mystery—a mystery at least equal to that of thinking itself.

It is worth noting that Hadamard is ever ready to accept as sidelights on his subject the reports of a Mozart or a Paul Valéry on their