ON SOME ASYMPTOTIC FORMULAS IN THE THEORY OF PARTITIONS

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Let $p(n)$ denote the number of unrestricted partitions of n. $p_k(n)$ denotes the number of partitions of n into precisely k summands, or what is the same into partitions whose largest summand is k. Auluck, Chowla and Gupta[1] announced the following conjecture:

For n fixed let $p_{k_0}(n)$ be the greatest $p_k(n)$; that is, $p_{k_0}(n) \geqq p_k(n)$. Then

$$k_0 \sim c^{-1}n^{1/2} \log n, \qquad c = \pi(2/3)^{1/2}. \tag{1}$$

They prove that

$$n^{1/2} < k_0 < (1 + \delta)c^{-1}n^{1/2} \log n$$

for every $\delta > 0$ if n is sufficiently large.

In the present note we shall prove (1). In fact we shall prove that

$$k_0 = c^{-1}n^{1/2} \log n + an^{1/2} + o(n^{1/2}) \quad \text{where} \quad c/2 = e^{-ca/2}. \tag{2}$$

They also conjectured that for $k_1 < k_2 \leqq k_0$, $p_{k_1}(n) \leqq p_{k_2}(n)$ and for $k_0 < k_1 < k_2$, $p_{k_1}(n) < p_{k_2}(n)$. They verify this conjecture for $n \leqq 32$. Recently Todd[2] published a table of all the $p_k(n)$ for $n \leqq 100$, and it is easy to verify the conjecture for $n \leqq 100$. I am unable to prove or disprove this conjecture. They also remark that $p_{k_0}(n)$ differs from $c^{-1}n^{1/2} \log n$ by less than 1 for $n \leqq 32$; (2) shows that for large n the difference tends to infinity.

Lehner and I[3] proved that if we denote

$$P_k(n) = \sum_{r \leqq k} p_r(n)$$

then for $k = c^{-1}n^{1/2} \log n + \lambda n^{1/2}$ we have the asymptotic formula

$$P_k(n)/p(n) = (1 + o(1)) \exp\left(-(2/c)e^{-cx/2}\right). \tag{3}$$

In proving (2) we shall use (3) a great deal, we shall also use the well known asymptotic formula

$$p(n) = (1 + o(1))(1/4 \cdot 3^{1/2}n) \exp(cn^{1/2}). \tag{4}$$

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[1] J. Indian Math. Soc. vol. 6 (1942) pp. 105–112.

[2] Proc. London Math. Soc. vol. 48 (1944) pp. 229–242.

[3] Duke Math. J. vol. 8 (1941) pp. 335–345.

Let $f(n)$ tend to infinity arbitrarily slowly; we easily obtain from (3) that for $k_1=[c^{-1}n^{1/2}\log n+f(n)n^{1/2}]$, $k_2=[c^{-1}n^{1/2}\log n-f(n)n^{1/2}]$,

$$(5)\qquad (1/p(n))(P_{k_1}(n)-P_{k_2}(n))\rightarrow 1 \quad \text{as} \quad n\rightarrow\infty.$$

We immediately obtain from (4) and (5) that for some $k_2<k_3<k_1$

$$(6)\qquad p_{k_3}(n)>c_1p(n)/n^{1/2}>(c_2/n^{3/2})\exp(cn^{1/2}).$$

$c_1, c_2, \cdots$ denote absolute constants. Thus

$$(7)\qquad p_{k_0}(n)\geqq p_{k_3}(n)>(c_2/n^{3/2})\exp(cn^{1/2}).$$

Now we show that for sufficiently large c_3

$$(8)\qquad k_0<c^{-1}n^{1/2}\log n+c_3n^{1/2}.$$

Let $k_4\geqq c^{-1}n^{1/2}\log n+c_3n^{1/2}$. It clearly follows from the definition of $p_k(n)$ and $P_k(n)$ that $p_{k_4}(n)=P_{k_4}(n-k_4)<p(n-k_4)$. Thus from (4)

$$\begin{aligned} p_{k_4}(n) &< (c_4/n)\exp(c(n-k_4)^{1/2}) < (c_4/n)\exp c(n^{1/2}-k_4/2n^{1/2}) \\ &< (c_4/n)\exp(c(n^{1/2}-\log n/2-c_3/2)) \\ &< (c_2/n^{3/2})\exp(cn^{1/2}) < p_{k_0}(n) \end{aligned}$$

for sufficiently large c_3, and this proves (8).

Next we prove that for sufficiently large c_5

$$(9)\qquad k_0>c^{-1}n^{1/2}\log n-c_5n^{1/2}.$$

Suppose (9) does not hold. We obtain from (7) that for some $k_0<c^{-1}n^{1/2}\log n-c_5n^{1/2}$

$$(10)\qquad p_{k_0}(n)>(c_2/n^{3/2})\exp(cn^{1/2}).$$

We shall show that (10) leads to a contradiction. First we show that

$$(11)\qquad p_k(n)\leqq p_{k+i}(n+j) \qquad \text{for } j\geqq i.$$

We have

$$(12)\qquad p_k(n)\leqq p_{k+i}(n+i)\leqq p_{k+i}(n+j).$$

The first inequality of (12) we obtain by mapping the partition $a_1+\cdots+k$ of $p_k(n)$ into $a_1+\cdots+(k+i)$ which belongs to $p_{k+i}(n+i)$, the second part we obtain by adding $j-i$ 1's to every partition of $p_{k+i}(n+i)$; this proves (11).

Put $[n^{1/2}]=b$; we have from (10) and (11) for $0\leqq i\leqq b$

$$\begin{aligned} p_{k_0+i}(n+b) &\geqq p_{k_0}(n) > (c_2/n^{3/2})\exp(cn^{1/2}) \\ &> (c_6/n^{3/2})\exp(c(n+b)^{1/2}). \end{aligned}$$

Thus

$$\sum_{i=0}^{b} p_{k_0+i}(n+b) > (c_6/n)\exp\left(c(n+b)^{1/2}\right). \tag{13}$$

Now we obtain from (5) that for every ϵ and sufficiently large c_5 and n

$$\sum_{k>k_0+b} p_k(n+b) > (1-\epsilon)p(n+b). \tag{14}$$

The proof of (14) follows immediately from the fact that $k_0+b < c^{-1}n^{1/2}\log n-(c_5-1)n^{1/2}$, thus (5) can be applied. From (13) and (14) we have

$$\begin{aligned} p(n+b) &> \sum_{i=0}^{b} p_{k_0+i}(n+b) + \sum_{k>k_0+b} p_k(n+b) \\ &> (1-\epsilon)p(n+b) + (c_{6/n})\exp\left(c(n+b)^{1/2}\right). \end{aligned}$$

Thus

$$\epsilon p(n+b) > (c_{6/n})\exp\left(c(n+b)^{1/2}\right),$$

which contradicts (4); this proves (9).

We now know from (8) and (9) that k_0 has to satisfy

$$c^{-1}n^{1/2}\log n - c_5 n^{1/2} < k_0 < c^{-1}n^{1/2}\log n + c_3 n^{1/2}.$$

Put

$$k_0 = c^{-1}n^{1/2}\log n + xn^{1/2}.$$

We obtain from (3) and (4) that

$$\begin{aligned} p_{k_0}(n) &= P_{k_0}(n-k_0) \\ &= (1+o(1))p(n)n^{-1/2}\exp\left(-cx/2-(2/c)\exp(-cx/2)\right).\text{[4]} \end{aligned} \tag{15}$$

The right side is maximal if $c/2=\exp(-cx/2)$, which completes the proof of (2).

We immediately obtain from (2) and (15) that

$$\lim p_{k_0}(n)n^{1/2}/p(n) = \exp\left(-ca/2-(2/c)\exp(-ax/2)\right).$$

It would be easy to sharpen the error term $o(n^{1/2})$ in (2) by getting an error term in (3), but it seems very hard to get a sufficiently good inequality to prove the conjecture of Auluck, Chowla and Gupta.

Denote by $Q(n)$ the number of partitions of n into unequal parts. $Q_k(n)$ denotes the number of partitions of n into precisely k unequal parts. Define k_0 by

$$Q_{k_0}(n) \geqq Q_k(n).$$

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[4] This formula is due to Auluck, Chowla and Gupta (ibid).

It has been conjectured that for $k_1 < k_2 \leqq k_0$, $Q_{k_1}(n) < Q_{k_2}(n)$ and for $k_0 < k_1 < k_2$, $Q_{k_1}(n) \geqq Q_{k_2}(n)$. This conjecture we can not decide. But by using Theorem 3.3 of our paper with Lehner we can show that

$$k_0 = 2 \log 2n^{1/2}/\pi(1/3)^{1/2} + dn^{1/4} + o(n^{1/4})$$

for a certain constant d. Also

$$\lim n^{1/4} Q_{k_0}(n)/Q(n) \to e, \quad \text{for a certain constant } e.$$

We do not discuss the proofs. They are similar but slightly more complicated than the proof of (2).

It would be interesting to get an asymptotic formula for $p_k(n)$ and $Q_k(n)$. Perhaps the first step would be to get an asymptotic formula for log $p_k(n)$. It is easy to see that for $k = o(n^{1/2})$

$$\log p_k(n) = o(n^{1/2})$$

and if $k/n^{1/2} \to \infty$

$$\log p_k(n)/\log p(n) \to 1.$$

The proofs can be obtained easily by simple Tauberian theorems.

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